Stochastic control under progressive enlargement of filtrations and applications to multiple defaults risk management[*]

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Abstract

We formulate and investigate a general stochastic control problem under a progressive enlargement of filtration. The global information is enlarged from a reference filtration and the knowledge of multiple random times together with associated marks when they occur. By working under a density hypothesis on the conditional joint distribution of the random times and marks, we prove a decomposition of the original stochastic control problem under the global filtration into classical stochastic control problems under the reference filtration, which are determined in a finite backward induction. Our method revisits and extends in particular stochastic control of diffusion processes with finite number of jumps. This study is motivated by optimization problems arising in default risk management, and we provide applications of our decomposition result for the indifference pricing of defaultable claims, and the optimal investment under bilateral counterparty risk. The solutions are expressed in terms of BSDEs involving only Brownian filtration, and remarkably without jump terms coming from the default times and marks in the global filtration.

Key words: stochastic control, progressive enlargement of filtrations, decomposition in the reference filtration, multiple default times, risk management.

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1 Introduction

The field of stochastic control has known important developments over these last years, inspired especially by various problems in economics and finance arising in risk management, option hedging, optimal investment, portfolio selection or real options valuation. A vast literature on this topic and its applications has grown with different approaches ranging from dynamic programming method, Hamilton-Jacobi-Bellman Partial Differential Equations (PDEs) and Backward Stochastic Differential Equations (BSDEs) to convex martingale duality methods. We refer to the monographs [7], [26], [18] or [19] for recent updates on this subject. In particular, the theory of BSDEs has emerged as a major research topic with original and significant contributions related to stochastic control and its financial applications, see a recent overview in [5].

On the other hand, the field of enlargement of filtrations is a traditional subject in probability theory initiated by fundamental works of the French school in the 80s, see e.g. [11], [9], [12], and the recent lecture notes [16]. It knows a renewed interest due to its natural application in credit risk research where it appears as a powerful tool for modelling default events. For an overview, we refer to the books [3], [4], [23] or the lecture notes [2]. The standard approach of credit event is based on the enlargement of a reference filtration $\mathbb{F}$ (the default-free information structure) by the knowledge of a default time when it occurs, leading to the global filtration $\mathbb{G}$, and called progressive enlargement of filtrations. Moreover, it assumes that the credit event should arrive by surprise, i.e. it is a totally inaccessible random time for the reference filtration. Hence, the main approaches consist in modelling the intensity of the random time (usually referred to as the reduced-form approach), or more generally in the modelling of the conditional law of this random time, and referred to as density hypothesis, see [6]. The stability of the class of semimartingale, usually called **(H')** hypothesis, and meaning that any $\mathbb{F}$-semimartingale remains a $\mathbb{G}$-semimartingale, is a fundamental property both in probability and finance where it is closely related to the absence of arbitrage. It holds true under the density hypothesis, and the related canonical decomposition in the enlarged filtration can be explicitly expressed, as shown in [10]. A stronger assumption than **(H')** hypothesis is the so-called immersion property or **(H)** hypothesis, denoting the fact that $\mathbb{F}$-martingales remain $\mathbb{G}$-martingales.

The purpose of this paper is to combine both features of stochastic control and progressive enlargement of filtrations in view of applications in finance, in particular for defaults risk management. We formulate and study the general structure for such control problems by considering a progressive enlargement with multiple random times and associated marks. These marks represent for example in credit events jump sizes of asset values, which may arrive several times by surprise and cannot be predicted from the past observation of asset processes. We work under the density hypothesis on the conditional joint distribution of the random times and marks. Our new approach consist in decomposing the initial control problem in the $\mathbb{G}$-filtration into a finite sequence of control problems formulated in the $\mathbb{F}$-filtration, and which are determined recursively. This is based on an enlightening representation of any $\mathbb{G}$-predictable or optional process that we split into indexed $\mathbb{F}$-predictable or optional processes between each random time. This point of view allows

us to change of regimes in the state process, and to modify the control set and the gain functions between random times. This flexibility in the formulation of the stochastic control problem appears also quite useful and relevant for financial interpretation. Our method consist basically in projecting $\mathbb{G}$-processes into the reference $\mathbb{F}$-filtration between two random times, and features some similarities with filtering approach. This contrasts with the standard approach in progressive enlargement of filtration focusing on the representation of controlled state process in the $\mathbb{G}$-filtration where the control set has to be fixed at the initial time. Moreover, in this global approach, one usually assumes that **(H)** hypothesis holds in order to get a martingale representation in the $\mathbb{G}$-filtration. In this case, the solution is then characterized from dynamic programming method in the $\mathbb{G}$-filtration via PDEs with integrodifferential terms or BSDEs with jumps. By means of our $\mathbb{F}$-decomposition result under the density hypothesis (and without assuming **(H)** hypothesis), we can solve each stochastic control problem by dynamic programming in the $\mathbb{F}$-filtration, which leads typically to PDEs or BSDEs related only to Brownian motion, thus simpler a priori than Integro-PDEs and BSDEs with jumps. Our decomposition method revisits and more importantly extends stochastic control of diffusion processes with finite number of jumps, and gives some new insight for studying Integro-PDEs and BSDEs with jumps. We illustrate our methodology with two financial applications in default risk management. The first one considers the problem of indifference pricing of defaultable claims, and the second application deals with an optimal investment problem under bilateral contagion risk with two nonordered default times. The solutions are explicitly expressed in terms of BSDEs involving only Brownian motion.

The paper is organized as follows. The next section presents the general framework of progressive enlargement of filtration with successive random times and marks. We state the decomposition result for a $\mathbb{G}$-predictable and optional process, and as a consequence we derive under the density hypothesis the computation of expectation functionals of $\mathbb{G}$-optional processes in terms of $\mathbb{F}$-expectations. In Section 3, we formulate the abstract stochastic control problem in this context and connect it in particular to diffusion processes with jumps. Section 4 contains the main $\mathbb{F}$-decomposition result of the initial stochastic control problem. The case of enlargement of filtration with multiple (and not necessarily successive) random times is considered in Section 5, and we show how to derive the results from the case of successive random times with auxiliary marks. Finally, Section 6 is devoted to some applications in risk management, where we present the results and postpone the detailed proofs and more examples in a forthcoming paper [13].

2   Progressive enlargement of filtration with successive random times

We fix a probability space $(\Omega, \mathcal{G}, \mathbb{P})$, and we start with a reference filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions ($\mathcal{F}_0$ contains the null sets of $\mathbb{P}$ and $\mathbb{F}$ is right continuous: $\mathcal{F}_t = \mathcal{F}_{t^+} := \cap_{s>t} \mathcal{F}_s$). We consider a vector of n random times $\tau_1, \ldots, \tau_n$ (i.e. nonnegative $\mathcal{G}$-random variables) and a vector of n $\mathcal{G}$-measurable random variables $\zeta_1, \ldots, \zeta_n$ valued in some Borel subset E of $\mathbb{R}^m$. The default information is the knowledge of these default

times τ_k when they occur, together with the associated marks ζ_k. For each $k = 1, \ldots, n$, it is defined in mathematical terms as the smallest right-continuous filtration $\mathbb{D}^k = (\mathcal{D}_t^k)_{t \geq 0}$ such that τ_k is a $\mathbb{D}^k$-stopping time, and ζ_k is $\mathcal{D}_{\tau_k}^k$-measurable. In other words, $\mathcal{D}_t^k = \tilde{\mathcal{D}}_{t+}^k$, where $\tilde{\mathcal{D}}_t^k = \sigma(\zeta_k 1_{\tau_k \leq s}, s \leq t) \vee \sigma(1_{\tau_k \leq s}, s \leq t)$. The global market information is then defined by the progressive enlargement of filtration $\mathbb{G} = \mathbb{F} \vee \mathbb{D}^1 \vee \ldots \vee \mathbb{D}^n$. The filtration $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ is the smallest filtration containing $\mathbb{F}$, and such that for any $k = 1, \ldots, n$, τ_k is a $\mathbb{G}$-stopping time, and ζ_k is $\mathcal{G}_{\tau_k}$-measurable. With respect to the classical framework of progressive enlargement of filtration with a single random time extensively studied in the literature, we consider here multiple random times together with marks. For simplicity of presentation, we first consider the case where the random times are ordered, i.e. $\tau_1 \leq \ldots \leq \tau_n$, and so valued in Δ_n on $\{\tau_n < \infty\}$, where

$$\Delta_k = \left\{ (\theta_1, \ldots, \theta_k) \in (\mathbb{R}_+)^k : \theta_1 \leq \ldots \leq \theta_k, \right\}, \quad k = 1, \ldots, n.$$

This means actually that the observations of interest are the ranked default times (together with the marks). We shall indicate in Section 5 how to adapt the results in the case of multiple random times not necessarily ordered.

We introduce some notations used throughout the paper.

- $\mathcal{P}(\mathbb{F})$ (resp. $\mathcal{P}(\mathbb{G})$) is the σ-algebra of $\mathbb{F}$ (resp. $\mathbb{G}$)-predictable measurable subsets on $\mathbb{R}_+ \times \Omega$, i.e. the σ-algebra generated by the left-continuous $\mathbb{F}$-adapted (resp. $\mathbb{G}$-adapted) processes. We also let $\mathcal{P}_\mathbb{F}$ (resp. $\mathcal{P}_\mathbb{G}$) denote the set of processes that are $\mathbb{F}$-predictable (resp. $\mathbb{G}$-predictable), i.e. $\mathcal{P}(\mathbb{F})$-measurable (resp. $\mathcal{P}(\mathbb{G})$-measurable).

- $\mathcal{O}(\mathbb{F})$ (resp. $\mathcal{O}(\mathbb{G})$) is the σ-algebra of $\mathbb{F}$ (resp. $\mathbb{G}$)-optional measurable subsets on $\mathbb{R}_+ \times \Omega$, i.e. the σ-algebra generated by the right-continuous $\mathbb{F}$-adapted (resp. $\mathbb{G}$-adapted) processes. We also let $\mathcal{O}_\mathbb{F}$ (resp. $\mathcal{O}_\mathbb{G}$) denote the set of processes that are $\mathbb{F}$-optional (resp. $\mathbb{G}$-optional), i.e. $\mathcal{O}(\mathbb{F})$-measurable (resp. $\mathcal{O}(\mathbb{G})$-measurable).

- For $k = 1, \ldots, n$, we denote by $\mathcal{P}_\mathbb{F}^k(\Delta_k, E^k)$ (resp. $\mathcal{O}_\mathbb{F}^k(\Delta_k, E^k)$) the set of of indexed processes $Y^k(.)$ such that the map $(t, \omega, \theta_1, \ldots, \theta_k, e_1, \ldots, e_k) \to Y_t^k(\omega, \theta_1, \ldots, \theta_k, e_1, \ldots, e_k)$ is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$-measurable (resp. $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$-measurable).

- For $\theta = (\theta_1, \ldots, \theta_n) \in \Delta_n$, $e = (e_1, \ldots, e_n) \in E^n$, we denote by

$$\theta^{(k)} = (\theta_1, \ldots, \theta_k), \qquad e^{(k)} = (e_1, \ldots, e_k), \quad k = 1, \ldots, n.$$

The following result provides the key decomposition of predictable and optional processes with respect to this progressive enlargement of filtration. This extends a classical result, see e.g. Lemma 4.4 in [11] or Chapter 6 in [21], stated for a progressive enlargement of filtration with a single random time.

Lemma 2.1 *Any $\mathbb{G}$-predictable process $Y = (Y_t)_{t \geq 0}$ is represented as*

$$
\begin{aligned}
Y_t &= Y_t^0 1_{t \leq \tau_1} + \sum_{k=1}^{n-1} Y_t^k(\tau_1, \ldots, \tau_k, \zeta_1, \ldots, \zeta_k) 1_{\tau_k < t \leq \tau_{k+1}} \\
&\quad + Y_t^n(\tau_1, \ldots, \tau_n, \zeta_1, \ldots, \zeta_n) 1_{\tau_n < t}, \quad t \geq 0,
\end{aligned}
\tag{2.1}
$$

where $Y^0 \in \mathcal{P}_{\mathbb{F}}$, and $Y^k \in \mathcal{P}_{\mathbb{F}}^k(\Delta_k, E^k)$, for $k = 1, \ldots, n$. Any $\mathbb{G}$-optional process $Y = (Y_t)_{t \geq 0}$ is represented as

$$
\begin{aligned}
Y_t \;=\; & Y_t^0 1_{t < \tau_1} + \sum_{k=1}^{n-1} Y_t^k(\tau_1, \ldots, \tau_k, \zeta_1, \ldots, \zeta_k) 1_{\tau_k \leq t < \tau_{k+1}} \\
& + Y_t^n(\tau_1, \ldots, \tau_n, \zeta_1, \ldots, \zeta_n) 1_{\tau_n \leq t}, \quad t \geq 0,
\end{aligned}
\tag{2.2}
$$

where $Y^0 \in \mathcal{O}_{\mathbb{F}}$, and $Y^k \in \mathcal{O}_{\mathbb{F}}^k(\Delta_k, E^k)$, for $k = 1, \ldots, n$.

Proof. We prove the decomposition result for predictable processes by induction on n. We denote by $\mathbb{G}^n = \mathbb{F} \vee \mathbb{D}^1 \vee \ldots \vee \mathbb{D}^n$.

Step 1. Suppose first that $n = 1$, so that $\mathbb{G} = \mathbb{F} \vee \mathbb{D}^1$. Let us consider generators of $\mathcal{P}(\mathbb{G})$, which are processes in the form

$$
Y_t \;=\; f_s \, g(\zeta_1 1_{\tau_1 \leq s}) h(\tau_1 \wedge s) 1_{t > s}, \quad t \geq 0,
$$

with $s \geq 0$, f_s $\mathcal{F}_s$-measurable, g measurable defined on $E \cup \{0\}$, and h measurable defined on $\mathbb{R}_+$. By taking

$$
Y_t^0 \;=\; f_s \, g(0) h(s) 1_{t > s}, \quad \text{and} \quad Y_t^1(\theta_1, e) \;=\; f_s g(e 1_{\theta_1 \leq s}) h(\theta_1 \wedge s) 1_{t > s},
$$

we see that the decomposition (2.1) holds for generators of $\mathcal{P}(\mathbb{G})$. We then extend this decomposition for any $\mathcal{P}(\mathbb{G})$-measurable processes, by the monotone class theorem.

Step 2. Suppose that the result holds for n, and consider the case with $n+1$ ranked default times, so that $\mathbb{G} = \mathbb{G}^n \vee \mathbb{D}^{n+1}$, $\mathcal{D}_t^{n+1} = \tilde{\mathcal{D}}_{t+}^{n+1}$, where $\tilde{\mathcal{D}}_t^{n+1} = \sigma(\zeta_{n+1} 1_{\tau_{n+1} \leq s}, s \leq t) \vee \sigma(1_{\tau_{n+1} \leq s}, s \leq t)$. By the same arguments of enlargement of filtration with one default time as in Step 1, we derive that any $\mathcal{P}(\mathbb{G})$-measurable process Y is represented as

$$
Y_t \;=\; Y_t^{0,(n)} 1_{t \leq \tau_{n+1}} + Y_t^{1,(n)}(\tau_{n+1}, \zeta_{n+1}) 1_{\tau_{n+1} < t},
\tag{2.3}
$$

where $Y^{0,(n)}$ is $\mathcal{P}(\mathbb{G}^n)$-measurable, and $(t, \omega, \theta_{n+1}, e_{n+1}) \mapsto Y_t^{1,(n)}(\omega, \theta_{n+1}, e_{n+1})$ is $\mathcal{P}(\mathbb{G}^n) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(E)$-measurable. Now, from the induction hypothesis for $\mathbb{G}^n$, we have

$$
\begin{aligned}
Y_t^{0,(n)} \;=\; & Y_t^{0,0,(n)} 1_{t \leq \tau_1} + \sum_{k=1}^{n-1} Y_t^{k,0,(n)}(\tau_1, \ldots, \tau_k, \zeta_1, \ldots, \zeta_k) 1_{\tau_k < t \leq \tau_{k+1}} \\
& + Y_t^{n,0,(n)}(\tau_1, \ldots, \tau_n, \zeta_1, \ldots, \zeta_n) 1_{\tau_n < t}, \quad t \geq 0,
\end{aligned}
$$

where $Y^{0,0,(n)} \in \mathcal{P}_{\mathbb{F}}$, and $Y^{k,0,(n)} \in \mathcal{P}_{\mathbb{F}}^k(\Delta_k, E^k)$, for $k = 1, \ldots, n$. Similarly, we have

$$
\begin{aligned}
Y_t^{1,(n)}(\theta_{n+1}, e_{n+1}) \;=\; & Y_t^{0,1,(n)}(\theta_{n+1}, e_{n+1}) 1_{t \leq \tau_1} \\
& + \sum_{k=1}^{n-1} Y_t^{k,1,(n)}(\tau_1, \ldots, \tau_k, \zeta_1, \ldots, \zeta_k, \theta_{n+1}, e_{n+1}) 1_{\tau_k < t \leq \tau_{k+1}} \\
& + Y_t^{n,1,(n)}(\tau_1, \ldots, \tau_n, \zeta_1, \ldots, \zeta_n, \theta_{n+1}, e_{n+1}) 1_{\tau_n < t}, \quad t \geq 0,
\end{aligned}
$$

5

where $Y^{0,1,(n)} \in \mathcal{P}^1_{\mathbb{F}}(\mathbb{R}_+, E)$, $Y^{k,1,(n)} \in \mathcal{P}^{k+1}_{\mathbb{F}}(\Delta_k \times \mathbb{R}_+, E^{k+1})$, $k = 1, \ldots, n$. Finally, plugging these two decompositions with respect to $\mathcal{P}(\mathbb{G}^n)$ into relation (2.3), and recalling that $\tau_1 \leq \ldots \leq \tau_n \leq \tau_{n+1}$, we get the required decomposition at level $n+1$ for $\mathbb{G}$:

$$
\begin{aligned}
Y_t \;=\;& Y^{0,0,(n)}_t 1_{t \leq \tau_1} + \sum_{k=1}^{n} Y^{k,0,(n)}_t(\tau_1, \ldots, \tau_k, \zeta_1, \ldots, \zeta_k) 1_{\tau_k < t \leq \tau_{k+1}} \\
& + Y^{n+1}_t(\tau_1, \ldots, \tau_{n+1}, \zeta_1, \ldots, \zeta_{n+1}) 1_{\tau_{n+1} < t}, \quad t \geq 0,
\end{aligned}
$$

where we notice that the indexed process Y^{n+1} defined by $Y^{n+1}(\theta_1, \ldots, \theta_{n+1}, e_1, \ldots, e_{n+1})$ $:= Y^{n,1,(n)}(\theta_1, \ldots, \theta_n, e_1, \ldots, e_n, \theta_{n+1}, e_{n+1})$, lies in $\mathcal{P}^{n+1}_{\mathbb{F}}(\Delta_{n+1}, E^{n+1})$.

The decomposition result for $\mathbb{G}$-optional processes is proved similarly by induction and considering generators of $\mathcal{O}(\mathbb{G}^1)$, which are processes in the form

$$
Y_t \;=\; f_s \, g(\zeta_1 1_{\tau_1 \leq s}) h(\tau_1 \wedge s) 1_{t \geq s}, \quad t \geq 0,
$$

with $s \geq 0$, f_s $\mathcal{F}_s$-measurable, g measurable defined on $E \cup \{0\}$, and h measurable defined on $\mathbb{R}_+$. $\qquad\square$

In view of the decomposition (2.1) or (2.2), we can then identify any $Y \in \mathcal{P}_{\mathbb{G}}$ (resp. $\mathcal{O}_{\mathbb{G}}$) with an $n+1$-tuple $(Y^0, \ldots, Y^n) \in \mathcal{P}_{\mathbb{F}} \times \ldots \times \mathcal{P}^n_{\mathbb{F}}(\Delta_n, E^n)$ (resp. $\mathcal{O}_{\mathbb{F}} \times \ldots \times \mathcal{O}^n_{\mathbb{F}}(\Delta_n, E^n)$).

We now require a density hypothesis on the random times and their associated jumps by assuming that for any t, the conditional distribution of $(\tau_1, \ldots, \tau_n, \zeta_1, \ldots, \zeta_n)$ given $\mathcal{F}_t$ is absolutely continuous with respect to a positive measure $\lambda(d\theta)\eta(de)$ on $\mathcal{B}(\Delta_n) \otimes \mathcal{B}(E^n)$, with λ the Lebesgue measure $\lambda(d\theta) = d\theta_1 \ldots d\theta_n$, and η a product measure $\eta(de) = \eta_1(de_1) \ldots \eta_1(de_n)$ on $\mathcal{B}(E) \otimes \ldots \otimes \mathcal{B}(E)$. More precisely, we assume that there exists $\gamma \in \mathcal{O}^n_{\mathbb{F}}(\Delta_n, E^n)$ such that

$$
(\mathbf{DH}) \qquad \mathbb{P}\big[(\tau_1, \ldots, \tau_n, \zeta_1, \ldots, \zeta_n) \in d\theta de | \mathcal{F}_t\big] \;=\; \gamma_t(\theta_1, \ldots, \theta_n, e_1, \ldots, e_n)
$$
$$
d\theta_1 \ldots d\theta_n \, \eta_1(de_1) \ldots \eta_1(de_n), \quad a.s.
$$

Remark 2.1 In the particular case where γ is in the form $\gamma_t(\theta, e) = \varphi_t(\theta)\psi_t(e)$, the condition **(DH)** means that the random times $(\tau_1, \ldots, \tau_n)$ and the jump sizes $(\zeta_1, \ldots, \zeta_n)$ are independent given $\mathcal{F}_t$, for all $t \geq 0$, and

$$
\mathbb{P}\big[(\tau_1, \ldots, \tau_n) \in d\theta | \mathcal{F}_t\big] \;=\; \varphi_t(\theta)\lambda(d\theta), \qquad \mathbb{P}\big[(\zeta_1, \ldots, \zeta_n) \in de | \mathcal{F}_t\big] \;=\; \psi_t(e)\eta(de), \quad a.s.
$$

This condition extends the usual density hypothesis for a random time in the theory of initial or progressive enlargement of filtration, see [9] or [10]. An important result in the theory of enlargement of filtration under the density hypothesis is the semimartingale invariance property, also called **(H')** hypothesis, i.e. any $\mathbb{F}$-semimartingale remains a $\mathbb{G}$-semimartingale. This result is related in finance to no-arbitrage conditions, and is thus also a desirable property from a economical viewpoint. Random times satisfying the density hypothesis are very well suitable for the analysis of credit risk events, as shown recently in [6]. We also refer to this paper for a discussion on the relation between the density hypothesis and the reduced-form (or intensity) approach in credit risk modelling.

In the sequel, it is useful to introduce the following notations. We denote by γ^0 the $\mathbb{F}$-optional process defined by

$$
\begin{aligned}
\gamma_t^0 &= \mathbb{P}\left[\tau_1 > t \mid \mathcal{F}_t\right] \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2.4)\\
&= \int_{E^n} \int_t^\infty \int_{\theta_1}^\infty \cdots \int_{\theta_{n-1}}^\infty \gamma_t(\theta_1, \ldots, \theta_n, e_1, \ldots, e_n) d\theta_n \ldots d\theta_1 \eta_1(de_1) \ldots \eta_n(de_n),
\end{aligned}
$$

and we denote by γ^k, $k = 1, \ldots, n-1$, the indexed process in $\mathcal{O}_{\mathbb{F}}^k(\Delta_k, E^k)$ defined by

$$
\begin{aligned}
&\gamma_t^k(\theta_1, \ldots, \theta_k, e_1, \ldots, e_k) \\
&= \int_{E^{n-k}} \int_t^\infty \int_{\theta_{k+1}}^\infty \cdots \int_{\theta_{n-1}}^\infty \gamma_t(\theta_1, \ldots, \theta_n, e_1, \ldots, e_n) d\theta_n \ldots d\theta_{k+1} \eta_1(de_{k+1}) \ldots \eta_1(de_n),
\end{aligned}
$$

so that for $k = 1, \ldots, n-1$,

$$
\mathbb{P}\left[\tau_{k+1} > t \mid \mathcal{F}_t\right] = \int_{E^k} \int_{\Delta_k} \gamma_t^k(\theta_1, \ldots, \theta_k, e_1, \ldots, e_k) d\theta_1 \ldots d\theta_k \eta_1(de_1) \ldots \eta_1(de_k). \qquad (2.5)
$$

Notice that the family of measurable maps γ_k, $k = 0, \ldots, n$ can be also written in backward induction by

$$
\gamma_t^k(\theta_1, \ldots, \theta_k, e_1, \ldots, e_k) = \int_E \int_t^\infty \gamma_t^{k+1}(\theta_1, \ldots, \theta_{k+1}, e_1, \ldots, e_{k+1}) d\theta_{k+1} \eta_1(de_{k+1}),
$$

for $k = 0, \ldots, n-1$, starting from $\gamma^n = \gamma$. In view of (2.4)-(2.5), the process γ_k may be interpreted as the survival density process of τ_{k+1}, $k = 0, \ldots, n-1$.

The next result provides the computation for the optional projection of a $\mathcal{O}(\mathbb{G})$-measurable process on the reference filtration $\mathbb{F}$.

Lemma 2.2 *Let $Y = (Y^0, \ldots, Y^n)$ be a nonnegative (or bounded) $\mathbb{G}$-optional process. Then for any $t \geq 0$, we have*

$$
\begin{aligned}
\hat{Y}_t^{\mathbb{F}} &:= \mathbb{E}\left[Y_t \mid \mathcal{F}_t\right] \\
&= Y_t^0 \gamma_t^0 + \sum_{k=1}^n \int_{E^k} \int_0^t \cdots \int_{\theta_{k-1}}^t Y_t^k(\theta_1, \ldots, \theta_k, e_1, \ldots, e_k) \\
&\qquad\qquad\qquad\qquad \gamma_t^k(\theta_1, \ldots, \theta_k, e_1, \ldots, e_k) d\theta_k \ldots d\theta_1 \eta_1(de_1) \ldots \eta_1(de_k),
\end{aligned}
$$

where we used the convention that $\theta_{k-1} = 0$ for $k = 1$ in the above integral. Equivalently, we have the backward induction formula for $\hat{Y}_t^{\mathbb{F}} = \hat{Y}_t^{0,\mathbb{F}}$, where the $\hat{Y}_t^{k,\mathbb{F}}$ are given for any $t \geq 0$, by

$$
\begin{aligned}
\hat{Y}_t^{n,\mathbb{F}}(\theta, e) &= Y_t^n(\theta, e) \gamma_t(\theta, e) \\
\hat{Y}_t^{k,\mathbb{F}}(\theta^{(k)}, e^{(k)}) &= Y_t^k(\theta^{(k)}, e^{(k)}) \gamma_t^k(\theta^{(k)}, e^{(k)}) \\
&\qquad + \int_{\theta_k}^t \int_E \hat{Y}_t^{k+1,\mathbb{F}}(\theta^{(k)}, \theta_{k+1}, e^{(k)}, e_{k+1}) \eta_1(de_{k+1}) d\theta_{k+1},
\end{aligned}
$$

for $\theta = (\theta_1, \ldots, \theta_n) \in \Delta_n \cap [0, t]^n$, $e = (e_1, \ldots, e_n) \in E^n$.

Proof. Let $Y = (Y^0, \ldots, Y^n)$ be a nonnegative (or bounded) $\mathbb{G}$-optional process, decomposed as in (2.2) so that:

$$\mathbb{E}\big[Y_t \big| \mathcal{F}_t\big] = \mathbb{E}\Big[Y_t^0 1_{t<\tau_1} \Big| \mathcal{F}_t\Big] + \sum_{k=1}^{n} \mathbb{E}\Big[Y_t^k(\tau_1, \ldots, \tau_k, \zeta_1, \ldots, \zeta_k) 1_{\tau_k \le t < \tau_{k+1}} \Big| \mathcal{F}_t\Big], \quad (2.6)$$

with the convention that $\tau_{n+1} = \infty$. Now, for any $k = 1, \ldots, n$, we have under the density hypothesis **(DH)**

$$\mathbb{E}\Big[Y_t^k(\tau_1, \ldots, \tau_k, \zeta_1, \ldots, \zeta_k) 1_{\tau_k \le t < \tau_{k+1}} \Big| \mathcal{F}_t\Big]$$

$$= \int_{\Delta_n \times E^n} Y_t^k(\theta_1, \ldots, \theta_k, e_1, \ldots, e_k) 1_{\theta_k \le t < \theta_{k+1}} \gamma_t(\theta_1, \ldots, \theta_n, e_1, \ldots, e_n) \lambda(d\theta) \eta(de)$$

$$= \int_{E^k} \int_0^t \cdots \int_{\theta_{k-1}}^t Y_t^k(\theta_1, \ldots, \theta_k, e_1, \ldots, e_k) \gamma_t^k(\theta_1, \ldots, \theta_k, e_1, \ldots, e_k) d\theta_k \ldots d\theta_1$$

$$\eta_1(de_1) \ldots \eta_1(de_k),$$

where the second inequality follows from Fubini's theorem and the definition of γ^k. We also have

$$\mathbb{E}\Big[Y_t^0 1_{t<\tau_1} \Big| \mathcal{F}_t\Big] = Y_t^0 \mathbb{P}[\tau_1 > t | \mathcal{F}_t] = Y_t^0 \gamma_t^0.$$

We then get the required result by plugging these two last relations into (2.6). Finally, the backward formula for the $\mathbb{F}$-optional projection of Y is obtained by a straightforward induction. $\qquad\square$

As a consequence of the above backward induction formula for the optional projection, we derive a bakward formula for the computation of expectation functionals of $\mathbb{G}$-optional processes, which involves only $\mathbb{F}$-expectations.

Proposition 2.1 *Let* $Y = (Y^0, \ldots, Y^n)$ *and* $Z = (Z^0, \ldots, Z^n)$ *be two nonnegative (or bounded)* $\mathbb{G}$*-optional processes, and fix* $T \in (0, \infty)$.
The expectation $\mathbb{E}[\int_0^T Y_t dt + Z_T]$ *can be computed in a backward induction as*

$$\mathbb{E}\Big[\int_0^T Y_t dt + Z_T \Big] = J_0$$

where the J_k, $k = 0, \ldots, n$ *are given by*

$$J_n(\theta, e) = \mathbb{E}\Big[\int_{\theta_n}^T Y_t^n \gamma_t(\theta, e) dt + Z_T^n \gamma_T(\theta, e) \Big| \mathcal{F}_{\theta_n} \Big]$$

$$J_k(\theta^{(k)}, e^{(k)}) = \mathbb{E}\Big[\int_{\theta_k}^T Y_t^k \gamma_t^k(\theta^{(k)}, e^{(k)}) dt + Z_T^k \gamma_T^k(\theta^{(k)}, e^{(k)})$$

$$+ \int_{\theta_k}^T \int_E J_{k+1}(\theta^{(k)}, \theta_{k+1}, e^{(k)}, e_{k+1}) \eta_1(de_{k+1}) d\theta_{k+1} \Big| \mathcal{F}_{\theta_k} \Big],$$

for $\theta = (\theta_1, \ldots, \theta_n) \in \Delta_n \cap [0,T]^n$, $e = (e_1, \ldots, e_n) \in E^n$, *with the convention* $\theta_0 = 0$.

8

Proof. For any $\theta = (\theta_1, \ldots, \theta_n) \in \Delta_n \cap [0,T]^n$, $e = (e_1, \ldots, e_n) \in E^n$, let us define

$$J_k(\theta^{(k)}, e^{(k)}) = \mathbb{E}\Big[\int_{\theta_k}^T \hat{Y}_t^{k,\mathbb{F}}(\theta^{(k)}, e^{(k)})dt + \hat{Z}_T^{k,\mathbb{F}}(\theta^{(k)}, e^{(k)})\big|\mathcal{F}_{\theta_k}\Big],$$

where the $\hat{Y}^{k,\mathbb{F}}$ and $\hat{Z}^{k,\mathbb{F}}$ are defined in Lemma 2.2, associated respectively to Y and Z. Then $J_0 = \mathbb{E}[\int_0^T Y_t dt + Z_T]$, and we see from the backward induction for $\hat{Y}^{k,\mathbb{F}}$ and $\hat{Z}^{k,\mathbb{F}}$ that the J_k, $k = 0, \ldots, n$, satisfy

$$J_n(\theta, e) = \mathbb{E}\Big[\int_{\theta_n}^T Y_t^n \gamma_t(\theta, e)dt + Z_T^n(\theta, e)\gamma_T(\theta, e)\Big|\mathcal{F}_{\theta_n}\Big]$$

$$
\begin{aligned}
J_k(\theta^{(k)}, e^{(k)}) = &\ \mathbb{E}\Big[\int_{\theta_k}^T Y_t^k \gamma_t^k(\theta^{(k)}, e^{(k)})dt + Z_T^k \gamma_T^k(\theta^{(k)}, e^{(k)}) \\
&+ \int_{\theta_k}^T \int_{\theta_k}^t \int_E \hat{Y}_t^{k+1,\mathbb{F}}(\theta^{(k)}, \theta_{k+1}, e^{(k)}, e_{k+1})\eta_1(de_{k+1})d\theta_{k+1}dt \\
&+ \int_{\theta_k}^T \int_E \hat{Z}_T^{k+1,\mathbb{F}}(\theta^{(k)}, \theta_{k+1}, e^{(k)}, e_{k+1})\eta_1(de_{k+1})d\theta_{k+1}\Big|\mathcal{F}_{\theta_k}\Big] \\
= &\ \mathbb{E}\Big[\int_{\theta_k}^T Y_t^k \gamma_t^k(\theta^{(k)}, e^{(k)})dt + Z_T^k \gamma_T^k(\theta^{(k)}, e^{(k)}) \\
&+ \int_{\theta_k}^T \int_E \int_{\theta_{k+1}}^T \hat{Y}_t^{k+1,\mathbb{F}}(\theta^{(k)}, \theta_{k+1}, e^{(k)}, e_{k+1})dt\, \eta_1(de_{k+1})d\theta_{k+1} \\
&+ \int_{\theta_k}^T \int_E \hat{Z}_T^{k+1,\mathbb{F}}(\theta^{(k)}, \theta_{k+1}, e^{(k)}, e_{k+1})\eta_1(de_{k+1})d\theta_{k+1}\Big|\mathcal{F}_{\theta_k}\Big] \\
= &\ \mathbb{E}\Big[\int_{\theta_k}^T Y_t^k \gamma_t^k(\theta^{(k)}, e^{(k)})dt + Z_T^k \gamma_T^k(\theta^{(k)}, e^{(k)}) \\
&+ \int_{\theta_k}^T \int_E J_{k+1}(\theta^{(k)}, \theta_{k+1}, e^{(k)}, e_{k+1})\, \eta_1(de_{k+1})d\theta_{k+1}\Big|\mathcal{F}_{\theta_k}\Big],
\end{aligned}
$$

where we used Fubini's theorem in the second equality for J_k, and the law of iterated condional expectations for the last equality. This proves the required induction formula for J_k, $k = 0, \ldots, n$. $\qquad\square$

3 Abstract stochastic control problem

In this section, we formulate the general stochastic control problem in the context of progressively enlargement of filtration with successive random times and marks.

3.1 Controls and state process

A control is a $\mathbb{G}$-predictable process $\alpha = (\alpha^0, \ldots, \alpha^n) \in \mathcal{P}_\mathbb{F} \times \ldots \times \mathcal{P}_\mathbb{F}^n(\Delta_n, E^n)$, where the α^k, $k = 0, \ldots, n$, are valued in some given Borel set A_k of an Euclidian space. We denote by $\mathcal{P}_\mathbb{F}^k(A_0)$ (resp. $\mathcal{P}_\mathbb{F}^k(\Delta_k, E^k; A_k)$, $k = 1, \ldots, n$), the set of elements in $\mathcal{P}_\mathbb{F}$ (resp. $\mathcal{P}_\mathbb{F}^k(\Delta, E^k)$, $k = 1, \ldots, n$) valued in A_0 (resp. A_k, $k = 1, \ldots, n$). We set $A = A_0 \times \ldots \times A_n$, and denote by $\mathcal{A}_\mathbb{G}$ the set of *admissible controls* as the product $\mathcal{A}_\mathbb{F}^0 \times \ldots \times \mathcal{A}_\mathbb{F}^n$, where $\mathcal{A}_\mathbb{F}^0$ (resp. $\mathcal{A}_\mathbb{F}^k$, k

9

$= 1, \ldots, n)$ is some separable metric space of $\mathcal{P}_{\mathbb{F}}(A_0)$ (resp. $\mathcal{P}_{\mathbb{F}}^k(\Delta_k, E^k; A_k)$, $k = 1, \ldots, n)$. The separability condition is required for measurability selection issue.

The description of the controlled state process is formulated as follows:
• *Controlled state process between default times:* we are given a collection of measurable mappings:

$$(x, \alpha^0) \in \mathbb{R}^d \times \mathcal{A}_{\mathbb{F}}^0 \longmapsto X^{0,x,\alpha^0} \in \mathcal{O}_{\mathbb{F}} \tag{3.1}$$

$$(x, \alpha^k) \in \mathbb{R}^d \times \mathcal{A}_{\mathbb{F}}^k \longmapsto X^{k,x,\alpha^k} \in \mathcal{O}_{\mathbb{F}}^k(\Delta_k, E^k), \quad k = 1, \ldots, n, \tag{3.2}$$

such that we have the initial data:

$$X_0^{0,x,\alpha^0} = x, \quad \forall x \in \mathbb{R}^d,$$

$$X_{\theta_k}^{k,\xi,\alpha^k}(\theta_1, \ldots, \theta_k, e_1, \ldots, e_k) = \xi, \quad \forall \xi \, \mathcal{F}_{\theta_k} - \text{measurable}, \; k = 1, \ldots, n.$$

• *Jumps of the controlled state process:* we are given a collection of maps Γ^k on $\mathbb{R}_+ \times \Omega \times \mathbb{R}^d \times A_{k-1} \times E$, for $k = 1, \ldots, n$, such that

$$(t, \omega, x, a, e) \mapsto \Gamma_t^k(\omega, x, a, e) \text{ is } \mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(A_{k-1}) \otimes \mathcal{B}(E) - \text{measurable}.$$

• *Global controlled state process:* the controlled state process is then given by the mapping

$$(x, \alpha = (\alpha^0, \ldots, \alpha^n)) \in \mathbb{R}^d \times \mathcal{A}_{\mathbb{G}} \longmapsto X^{x,\alpha} \in \mathcal{O}_{\mathbb{G}},$$

where $X^{x,\alpha}$ is the process equal to

$$\begin{aligned}
X_t^{x,\alpha} = {} & \bar{X}_t^0 1_{t < \tau_1} + \sum_{k=1}^{n-1} \bar{X}_t^k(\tau_1, \ldots, \tau_k, \zeta_1, \ldots, \zeta_k) 1_{\tau_k \le t < \tau_{k+1}} \\
& + \bar{X}_t^n(\tau_1, \ldots, \tau_n, \zeta_1, \ldots, \zeta_n) 1_{\tau_n \le t}, \quad t \ge 0,
\end{aligned} \tag{3.3}$$

with $(\bar{X}^0, \ldots, \bar{X}^n) \in \mathcal{O}_{\mathbb{F}} \times \ldots \times \mathcal{O}_{\mathbb{F}}^n(\Delta_n, E^n)$ given by

$$\bar{X}^0 = X^{0,x,\alpha^0}$$

$$\bar{X}^k(\theta_1, \ldots, \theta_k, e_1, \ldots, e_k) = X^{k, \Gamma_{\theta_k}^k(\bar{X}_{\theta_k}^{k-1}, \alpha_{\theta_k}^{k-1}, e_k), \alpha^k}(\theta_1, \ldots, \theta_k, e_1, \ldots, e_k),$$

for $k = 1, \ldots, n$.

The interpretation is the following. Between the time interval $\tau_k = \theta_k$ and $\tau_{k+1} = \theta_{k+1}$, $k = 0, \ldots, n-1$ (with the convention $\theta_0 = 0$), the state process $X = \bar{X}^k$ is controlled by α_k, which is based on the basic information $\mathbb{F}$, and the knowledge of the past jump times and marks $(\theta_1, \ldots, \theta_k, e_1, \ldots, e_k)$. Then, at time θ_{k+1}, there is a jump on the state process determined by the map Γ^{k+1}, which depends on the current state value, control and information, but also on a "nonpredictable" mark $\zeta_{k+1} = e_{k+1}$ at time θ_{k+1}:

$$X_{\tau_{k+1}} = \Gamma_{\tau_{k+1}}^{k+1}(X_{\tau_{k+1}^-}, \alpha_{\tau_{k+1}}^k, \zeta_{k+1}).$$

3.2 Typical controlled state process

In typical applications, the dynamics of $X^0 = X^{0,x,\alpha^0}$, $X^k = X^{k,x,\alpha^k}$, $k = 1, \ldots, n$, are governed by diffusion processes:

$$
\begin{aligned}
dX_t^0 &= b_t^0(X_t^0, \alpha_t^0)dt + \sigma_t^0(X_t^0, \alpha_t^0)dW_t, \quad t \geq 0 & (3.4) \\
dX_t^k &= b_t^k(X_t^k, \alpha_t^k, \theta_1, \ldots, \theta_k, e_1, \ldots, e_k)dt & (3.5) \\
&\quad + \sigma_t^k(X_t^k, \alpha_t^k, \theta_1, \ldots, \theta_k, e_1, \ldots, e_k)dW_t, \quad t \geq \theta_k,
\end{aligned}
$$

Here, W is a standard m-dimensional $(\mathbb{P}, \mathbb{F})$-Brownian motion, and $(t, \omega, x, a) \to b_t^0(\omega, x, a)$, $\sigma_t^0(\omega, x, a)$ are $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(A_0)$-measurable maps valued respectively in $\mathbb{R}^d$ and $\mathbb{R}^{d \times m}$, for $k = 1, \ldots, n$, the maps $(t, \omega, x, a, \theta_1, \ldots, \theta_k, e_1, \ldots, e_k) \to b_t^k(\omega, x, a, \theta_1, \ldots, \theta_k, e_1, \ldots, e_k)$, $\sigma_t^k(\omega, x, u, \theta_1, \ldots, \theta_k, e_1, \ldots, e_k)$ are $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(A_k) \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$-measurable valued respectively in $\mathbb{R}^d$ and $\mathbb{R}^{d \times m}$. To alleviate notations, we omitted in (3.5) the dependence of X^k, α^k in $(\theta_1, \ldots, \theta_k, e_1, \ldots, e_k)$. We make the linear growth and Lipschitz assumptions on the functions $x \to b_t^k(x, .)$, $\sigma^k(x, .)$, $k = 0, \ldots, n$, in order to ensure for all $(\theta_1, \ldots, \theta_k, e_1, \ldots, e_k) \in \Delta_k \times E^k$, the existence and uniqueness of a solution $X^k(\theta_1, \ldots, \theta_k, e_1, \ldots, e_k)$ to the sde (3.4), (3.5), given the controls and the initial conditions, and this indexed process X^k lies in $\mathcal{O}_{\mathbb{F}}^k(\Delta_k, E^k)$. The dependence of the coefficients b^k, σ^k on the past jump times $\theta_1, \ldots, \theta_k$, and marks $e_1, \ldots, e_k$, corresponds to change of regimes after each jump time, and may be interpreted in finance as rating upgrades or downgrades. Also, typical choice for the set of admissible controls $\mathcal{A}_{\mathbb{F}}^k$ is subset of indexed $\mathbb{F}$-predictable processes in L^p, $p \in [1, \infty)$, and the separability of $\mathcal{A}_{\mathbb{F}}^k$ follows from the separability of L^p, see the discussion in [24].

Connection with controlled jump-diffusion processes.
Consider the particular case where the sets of controls A_k are identical, equal to A, and let us define the mappings b and σ on $\mathbb{R}_+ \times \Omega \times \mathbb{R}^d \times A$ by:

$$
\begin{aligned}
b_t(x, a) &= b_t^0(x, a)1_{t \leq \tau_1} + \sum_{k=1}^{n-1} b_t^k(x, a, \tau_1, \ldots, \tau_k, \zeta_1, \ldots, \zeta_k)1_{\tau_k < t \leq \tau_{k+1}} \\
&\quad + b_t^n(x, a, \tau_1, \ldots, \tau_n, \zeta_1, \ldots, \zeta_n)1_{t > \tau_n}, \\
\sigma_t(x, a) &= \sigma_t^0(x, a)1_{t \leq \tau_1} + \sum_{k=1}^{n-1} \sigma_t^k(x, a, \tau_1, \ldots, \tau_k, \zeta_1, \ldots, \zeta_k)1_{\tau_k < t \leq \tau_{k+1}} \\
&\quad + \sigma_t^n(x, a, \tau_1, \ldots, \tau_n, \zeta_1, \ldots, \zeta_n)1_{t > \tau_n},
\end{aligned}
$$

and notice that the maps $(t, \omega, x, a) \to b_t(\omega, x, a)$, $\sigma_t(\omega, x, a)$ are $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(A)$-measurable. Denote also by δ the mapping on $\mathbb{R}_+ \times \Omega \times \mathbb{R}^d \times A \times E$:

$$
\delta_t(x, a, e) = \sum_{k=1}^{n-1} \left(\Gamma_t^k(x, a, e) - x\right)1_{\tau_k < t \leq \tau_{k+1}} + \left(\Gamma_t^n(x, a, e) - x\right)1_{t > \tau_n},
$$

which is $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(A) \otimes \mathcal{B}(E)$-measurable. Let us denote by $\mu(dt, de)$ the integer-valued random measure associated to the times τ_k and the marks ζ_k, $k = 1, \ldots, n$, which

11

is then given by

$$\mu([0,t] \times B) = \sum_{k \geq 1} 1_{\tau_n \leq t} 1_B(\zeta_k), \quad \forall t \geq 0, \ B \in \mathcal{B}(E).$$

The progressive enlarged filtration $\mathbb{G}$ can then be written also as: $\mathbb{G} = \mathbb{F} \vee \mathbb{F}^\mu$ where $\mathbb{F}^\mu$ is the right-continuous filtration generated by the integer-valued random measure μ. Now, since the semimartingale property is preserved under the density hypothesis for this progressive enlargement of filtration, (see [10]), the process W remains a semimartingale under $(\mathbb{P}, \mathbb{G})$ (with a canonical decomposition, which can be explicitly expressed in terms of the density). Then, we can write the dynamics of the state process $X = X^{x,\alpha}$ in (3.3) as a controlled jump-diffusion process under $(\mathbb{P}, \mathbb{G})$:

$$dX_t = b_t(X_t, \alpha_t)dt + \sigma_t(X_t, \alpha_t)dW_t + \int_E \delta_t(X_{t^-}, \alpha_t, e)\mu(dt, de).$$

However, notice that in the above $\mathbb{G}$-formulation, the process W is not in general a Brownian motion under $(\mathbb{P}, \mathbb{G})$, unless the so-called **(H)** immersion property is satisfied, i.e. the martingale property is preserved from $\mathbb{F}$ to $\mathbb{G}$, which corresponds to the particular case where the density satisfies: $\gamma_t(\theta, e) = \gamma_\theta(\theta, e)$ for $t \geq \theta$.

In the classical formulation by controlled jump-diffusion processes, one has to fix a control set A, which is invariant during the time horizon. Here, the more general formulation (3.3) allows us to consider different control sets A_k between two default times, and this may be relevant in practical applications. Moreover, we have a suitable decomposition of the coefficients and controlled state process between random times, which provides a natural interpretation in economics and finance.

3.3 Stochastic control problem

In the general framework for the controlled process in (3.3), let us formulate the objective function for the stochastic control problem on a finite horizon T. The terminal gain function is given by a nonnegative map G_T on $\Omega \times \mathbb{R}^d$ such that $(\omega, x) \mapsto G_T(\omega, x)$ is $\mathcal{G}_T \otimes \mathcal{B}(\mathbb{R}^d)$-measurable, and which may be represented as

$$\begin{aligned}
G_T(x) = {} & G_T^0(x)1_{T < \tau_1} + \sum_{k=1}^{n-1} G_T^k(x, \tau_1, \ldots, \tau_k, \zeta_1, \ldots, \zeta_k)1_{\tau_k \leq T < \tau_{k+1}} \\
& + G_T^n(x, \tau_1, \ldots, \tau_n, \zeta_1, \ldots, \zeta_n)1_{\tau_n \leq T},
\end{aligned}$$

where G_T^0 is $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}^d)$-measurable, and G_T^k is $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$-measurable, for $k = 1, \ldots, n$. The running gain function is given by a nonnegative map f on $\Omega \times \mathbb{R}^d \times A$ such that $(t, \omega, x, a) \mapsto f_t(\omega, x, a)$ is $\mathcal{O}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(A)$-measurable, and which may be decomposed as

$$\begin{aligned}
f_t(x, a) = {} & f_t^0(x, a_0)1_{t < \tau_1} + \sum_{k=1}^{n-1} f_t^k(x, a_k, \tau_1, \ldots, \tau_k, \zeta_1, \ldots, \zeta_k)1_{\tau_k \leq t < \tau_{k+1}} \\
& + f_t^n(x, a_n, \tau_1, \ldots, \tau_n, \zeta_1, \ldots, \zeta_n)1_{\tau_n \leq t},
\end{aligned}$$

for $a = (a_0, \ldots, a_n) \in A = A_0 \times \ldots \times A_n$, where f^0 is $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(A_0)$-measurable, and f^k is $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(A_k) \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$-measurable, for $k = 1, \ldots, n$.

The value function for the stochastic control problem is then defined by:

$$V_0(x) = \sup_{\alpha \in \mathcal{A}_\mathbb{G}} \mathbb{E}\Big[\int_0^T f_t(X_t^{x,\alpha}, \alpha_t) dt + G_T(X_T^{x,\alpha}) \Big], \quad x \in \mathbb{R}^d. \tag{3.6}$$

Remark 3.1 In the formulation (3.6) of our stochastic control problem, there is a change of regimes in the running and terminal gain after each default time. This is in the spirit of the recent concept of forward or progressive utility functions introduced in [17].

4 $\mathbb{F}$-decomposition of the stochastic control problem

In this section, we provide a decomposition of the value function for the stochastic control problem in the $\mathbb{G}$-filtration, defined in (3.6), that we formulate in a backward induction for value functions of stochastic control in the $\mathbb{F}$-filtration. To alleviate notations, we shall often omit in (3.2) the dependence of $X^{k,x}$ on α^k and $(\theta_1, \ldots, \theta_k, e_1, \ldots, e_k)$ when there is no ambiguity.

Theorem 4.1 *The value function V_0 is obtained from the backward induction formula:*

$$V_n(x, \theta, e) = \operatorname*{ess\,sup}_{\alpha^n \in \mathcal{A}_\mathbb{F}^n} \mathbb{E}\Big[\int_{\theta_n}^T f_t^n(X_t^{n,x}, \alpha_t^n, \theta, e)\gamma_t(\theta, e) dt$$
$$+ G_T^n(X_T^{n,x}, \theta, e)\gamma_T(\theta, e) \Big| \mathcal{F}_{\theta_n} \Big] \tag{4.1}$$

$$V_k(x, \theta^{(k)}, e^{(k)}) = \operatorname*{ess\,sup}_{\alpha^k \in \mathcal{A}_\mathbb{F}^k} \mathbb{E}\Big[\int_{\theta_k}^T f_t^k(X_t^{k,x}, \alpha_t^k, \theta^{(k)}, e^{(k)})\gamma_t^k(\theta^{(k)}, e^{(k)}) dt$$
$$+ G_T^k(X_T^{k,x}, \theta^{(k)}, e^{(k)})\gamma_T^k(\theta^{(k)}, e^{(k)})$$
$$+ \int_{\theta_k}^T \int_E V_{k+1}\big(\Gamma_{\theta_{k+1}}^{k+1}(X_{\theta_{k+1}}^{k,x}, \alpha_{\theta_{k+1}}^k, e_{k+1}), \theta^{(k)}, \theta_{k+1}, e^{(k)}, e_{k+1}\big)$$
$$\eta_1(de_{k+1}) d\theta_{k+1} \Big| \mathcal{F}_{\theta_k} \Big], \quad k = 0, \ldots, n-1, \tag{4.2}$$

for all $\theta = (\theta_1, \ldots, \theta_n) \in \Delta_n \cap [0, T]^n$, $e = (e_1, \ldots, e_n) \in E^n$, $x \in \mathbb{R}^d$.

Remark 4.1 Each step in the backward induction for the determination of the original value function V_0 leads to the formulation of a family of value functions associated to standard stochastic control problem in the $\mathbb{F}$-filtration. Indeed, at step n, $V_n(x, .)$ is a family of value functions parametrized by $(\theta_1, \ldots, \theta_n) \in \Delta_n$, $(e_1, \ldots, e_n) \in E^n$, and corresponding to the stochastic control problem after the last default at time θ_n, with a running gain function f_t^n and terminal gain function G_T^n on the controlled state process X^n in the $\mathbb{F}$-filtration, and weighted by the $\mathcal{O}(\mathbb{F})$-measurable process γ. Now, suppose that at step $k + 1$, we have determined the family of value functions $V_{k+1}(x, .)$, $(\theta_1, \ldots, \theta_{k+1}) \in \Delta_{k+1}$,

$(e_1, \ldots, e_{k+1}) \in E^{k+1}$, and denote by $\hat{V}_{k+1}$ the map on $\Omega \times \mathbb{R}^d \times A_k \times \Delta_{k+1} \times E^k$:

$$\hat{V}_{k+1}\big(x, a_k, \theta^{(k)}, \theta_{k+1}, e^{(k)}\big)$$
$$= \int_E V_{k+1}\big(\Gamma^{k+1}_{\theta_{k+1}}(x, a_k, e_{k+1}), \theta^{(k)}, \theta_{k+1}, e^{(k)}, e_{k+1}\big) \eta_1(de_{k+1}).$$

Then, the family of value functions at step k, representing the value for the stochastic control problem after k defaults, is computed from the stochastic control problem in the $\mathbb{F}$-filtration with the running gain function f_t^k and terminal gain function G_T^k weighted by the $\mathcal{O}(\mathbb{F})$-measurable random variable γ^k, and with the running gain function $\hat{V}_{k+1}$:

$$V_k(x) = \operatorname*{ess\,sup}_{\alpha^k \in \mathcal{A}^k_{\mathbb{F}}} \mathbb{E}\Big[\int_{\theta_k}^T f_t^k(X_t^{k,x}, \alpha_t^k)\gamma_t^k dt + G_T^k(X_T^{k,x})\gamma_T^k$$
$$+ \int_{\theta_k}^T \hat{V}_{k+1}(X_{\theta_{k+1}}^{k,x}, \alpha_{\theta_{k+1}}^k, \theta_{k+1}) d\theta_{k+1} \Big| \mathcal{F}_{\theta_k} \Big]. \tag{4.3}$$

Here, we omitted the dependence in $\theta^{(k)} = (\theta_1, \ldots, \theta_k)$, $e^{(k)} = (e_1, \ldots, e_k)$ to alleviate notations. The two first terms in the rhs of (4.3) represent the gain functional when there is no more default after the k-th one, while the last term represents the gain in the case when a $k+1$-th default would occur between the last one at time $\tau_k = \theta_k$ and the finite horizon T. Finally, the decomposition in Theorem 4.1 also shows that an optimal control for the global problem in the $\mathbb{G}$-filtration is obtained by a concatenation of optimal controls for each subproblems V_k in the $\mathbb{F}$-filtration.

Proof of Theorem 4.1.
Fix $x \in \mathbb{R}^d$, $\alpha = (\alpha^0, \ldots, \alpha^n) \in \mathcal{A}_{\mathbb{G}}$, and consider the controlled state process $X^{x,\alpha}$. By definition of $X^{x,\alpha}$ in (3.3), $G_T(.)$ and $f_t(.)$, observe that the $\mathcal{G}_T$-measurable random variable $G_T(X_T^{x,\alpha})$ is decomposed according to the $n+1$-tuple $(G_T^0(\bar{X}_T^0), \ldots, G_T^n(\bar{X}_T^n))$, and the $\mathbb{G}$-optional process $f_t(X_t^{x,\alpha}, \alpha_t)$ is decomposed as $(f_t^0(\bar{X}_t^0, \alpha_t^0), \ldots, f_t^n(\bar{X}_t^n, \alpha_t^n))$. Let us now define by backward induction the maps J_k, $k = 0, \ldots, n$ by

$$J_n(x, \theta, e, \alpha) = \mathbb{E}\Big[\int_{\theta_n}^T f_t^n(X_t^{n,x}, \alpha_t^n, \theta, e)\gamma_t(\theta, e)dt + G_T^n(X_T^{n,x}, \theta, e)\gamma_T(\theta, e) \Big| \mathcal{F}_{\theta_n} \Big]$$

$$J_k(x, \theta^{(k)}, e^{(k)}, \alpha) = \mathbb{E}\Big[\int_{\theta_k}^T f_t^k(X_t^{k,x}, \alpha_t^k, \theta^{(k)}, e^{(k)})\gamma_t^k(\theta^{(k)}, e^{(k)})dt$$
$$+ G_T^k(X_T^{k,x}, \theta^{(k)}, e^{(k)})\gamma_T^k(\theta^{(k)}, e^{(k)})$$
$$+ \int_{\theta_k}^T \int_E J_{k+1}\big(\Gamma^{k+1}_{\theta_{k+1}}(X_{\theta_{k+1}}^{k,x}, \alpha_{\theta_{k+1}}^k, e_{k+1}), \theta^{(k)}, \theta_{k+1}, e^{(k)}, e_{k+1}, \alpha\big)$$
$$\eta_1(de_{k+1})d\theta_{k+1} \Big| \mathcal{F}_{\theta_k} \Big], \tag{4.4}$$

for any $x \in \mathbb{R}^d$, $\theta = (\theta_1, \ldots, \theta_n) \in \Delta_n \cap [0, T]^n$, $e = (e_1, \ldots, e_n) \in E^n$, and $\alpha = (\alpha^0, \ldots, \alpha^n) \in \mathcal{A}^0_{\mathbb{F}} \times \ldots \times \mathcal{A}^n_{\mathbb{F}}$. Let us denote by $\bar{J}_k(\theta^{(k)}, e^{(k)}) = J_k(\bar{X}^k_{\theta_k}, \theta^{(k)}, e^{(k)}, \alpha)$, $k = 0, \ldots, n$, and observe by definition of $X^{x,\alpha}$ and $\bar{X}^k$ in (3.3) that $\bar{J}_k$ satisfy the backward induction

14

formula:

$$\bar{J}_n(\theta, e) = \mathbb{E}\Big[\int_{\theta_n}^T f_t^n(\bar{X}_t^n, \alpha_t^n, \theta, e)\gamma_t(\theta, e)dt + G_T^n(\bar{X}_T^n, \theta, e)\gamma_T(\theta, e)\Big|\mathcal{F}_{\theta_n}\Big]$$

$$\begin{aligned}
\bar{J}_k(\theta^{(k)}, e^{(k)}) = \ & \mathbb{E}\Big[\int_{\theta_k}^T f_t^k(\bar{X}_t^k, \alpha_t^k, \theta^{(k)}, e^{(k)})\gamma_t^k(\theta^{(k)}, e^{(k)})dt \\
& + G_T^k(\bar{X}_T^k, \theta^{(k)}, e^{(k)})\gamma_T^k(\theta^{(k)}, e^{(k)}) \\
& + \int_{\theta_k}^T \int_E \bar{J}_{k+1}(\theta^{(k)}, \theta_{k+1}, e^{(k)}, e_{k+1})\eta_1(de_{k+1})d\theta_{k+1}\Big|\mathcal{F}_{\theta_k}\Big].
\end{aligned}$$

Therefore, from Proposition 2.1, we have the equality:

$$\mathbb{E}\Big[\int_0^T f(X_t^{x,\alpha}, \alpha_t)dt + G_T(X_T^{x,\alpha})\Big] = \bar{J}_0 = J_0(x, \alpha). \tag{4.5}$$

Let us now define the value function processes:

$$V_k(x, \theta^{(k)}, e^{(k)}) := \operatorname*{ess\,sup}_{\alpha \in \mathcal{A}_{\mathbb{G}}} J_k(x, \theta^{(k)}, e^{(k)}, \alpha), \tag{4.6}$$

for $k = 0, \ldots, n$, $x \in \mathbb{R}^d$, and $\theta = (\theta_1, \ldots, \theta_n) \in \Delta_n \cap [0,T]^n$, $e = (e_1, \ldots, e_n) \in E^n$. First, observe that this definition for $k = 0$ is consistent with the definition of the value function V_0 of the stochastic control problem (3.6) from the relation (4.5). Then, it remains to prove that the value functions V_k defined in (4.6) satisfy the backward induction formula in the assertion of the theorem. For $k = n$, and since $J_n(x, \theta, e, \alpha)$ depends on α only through its last component α^n, the relation (4.1) holds true. Next, from the backward induction (4.4) for J_k, and the definition of V_{k+1}, we have for all $\alpha = (\alpha^0, \ldots, \alpha^n) \in \mathcal{A}_{\mathbb{G}}$:

$$\begin{aligned}
J_k(x, \theta^{(k)}, e^{(k)}, \alpha) \leq \ & \mathbb{E}\Big[\int_{\theta_k}^T f_t^k(X_t^{k,x}, \alpha_t^k, \theta^{(k)}, e^{(k)})\gamma_t^k(\theta^{(k)}, e^{(k)})dt \\
& + G_T^k(X_T^{k,x}, \theta^{(k)}, e^{(k)})\gamma_T^k(\theta^{(k)}, e^{(k)}) \\
& + \int_{\theta_k}^T \int_E V_{k+1}\big(\Gamma_{\theta_{k+1}}^{k+1}(X_{\theta_{k+1}}^{k,x}, \alpha_{\theta_{k+1}}^k, e_{k+1}), \theta^{(k)}, \theta_{k+1}, e^{(k)}, e_{k+1}\big) \\
& \qquad\qquad \eta_1(de_{k+1})d\theta_{k+1}\Big|\mathcal{F}_{\theta_k}\Big] \\
\leq \ & \bar{V}_k(x, \theta^{(k)}, e^{(k)}), \tag{4.7}
\end{aligned}$$

where $\bar{V}_k$ is defined by the rhs of (4.2). By taking the supremum over α in the inequality (4.7), this shows that $V_k \leq \bar{V}_k$. Conversely, fix $x \in \mathbb{R}^d$, $\theta = (\theta_1, \ldots, \theta_n) \in \Delta_n \cap [0,T]^n$, $e = (e_1, \ldots, e_n) \in E^n$, and let us prove that $V_k(x, \theta^{(k)}, e^{(k)}) \geq \bar{V}_k(x, \theta^{(k)}, e^{(k)})$. Fix an arbitrary $\alpha^k \in \mathcal{A}_{\mathbb{F}}^k$, and the associated controlled process $X^{k,x}$. By definition of V_{k+1}, for any $\omega \in \Omega$, $\varepsilon > 0$, there exists $\alpha^{\omega,\varepsilon} \in \mathcal{A}_{\mathbb{G}}$, which is an ε-optimal control for $V_{k+1}(., \theta^{(k)}, e^{(k)})$ at $(\omega, \Gamma_{\theta_{k+1}}^{k+1}(X_{\theta_{k+1}}^{k,x}, \alpha_{\theta_{k+1}}^k, e_{k+1}))$. Recalling that the set of admissible controls is a separable metric space, one can use a measurable selection result (see e.g. [25]) to find $\alpha^\varepsilon \in \mathcal{A}_{\mathbb{G}}$ s.t. $\alpha_t^\varepsilon(\omega) = \alpha_t^{\omega,\varepsilon}(\omega)$, $dt \otimes d\mathbb{P}$ a.e., and so

$$\begin{aligned}
& V_{k+1}\big(\Gamma_{\theta_{k+1}}^{k+1}(X_{\theta_{k+1}}^{k,x}, \alpha_{\theta_{k+1}}^k, e_{k+1}), \theta^{(k)}, \theta_{k+1}, e^{(k)}, e_{k+1}\big) - \varepsilon \\
\leq \ & J_{k+1}\big(\Gamma_{\theta_{k+1}}^{k+1}(X_{\theta_{k+1}}^{k,x}, \alpha_{\theta_{k+1}}^k, e_{k+1}), \theta^{(k)}, \theta_{k+1}, e^{(k)}, e_{k+1}, \alpha^\varepsilon\big), \qquad a.s.
\end{aligned}$$

15

Denote by $(\alpha^{\varepsilon,0}, \ldots, \alpha^{\varepsilon,n})$ the $n+1$-tuple associated to $\alpha^\varepsilon \in \mathcal{A}_\mathbb{G}$, and let us consider the admissible control $\tilde{\alpha}^\varepsilon = (\alpha^{\varepsilon,0}, \ldots, \alpha^k, \alpha^{\varepsilon,k+1}, \ldots, \alpha^{\varepsilon,n}) \in \mathcal{A}_\mathbb{G}$ consisting in substituting the k-th component of α^ε by $\alpha^k \in \mathcal{A}_\mathbb{F}^k$. Since $J_{k+1}(x, \theta, e, \alpha)$ depends on α only through its last components $(\alpha^{k+1}, \ldots, \alpha^n)$, we have from (4.4)

$$
\begin{aligned}
V_k(x, \theta^{(k)}, e^{(k)}) \;\geq\;& J_k(x, \theta^{(k)}, e^{(k)}, \tilde{\alpha}^\varepsilon) \\
=\;& \mathbb{E}\Big[\int_{\theta_k}^T f_t^k(X_t^{k,x}, \alpha_t^k, \theta^{(k)}, e^{(k)}) \gamma_t^k(\theta^{(k)}, e^{(k)}) dt \\
& + G_T^k(X_T^{k,x}, \theta^{(k)}, e^{(k)}) \gamma_T^k(\theta^{(k)}, e^{(k)}) \\
& + \int_{\theta_k}^T \int_E J_{k+1}\big(\Gamma_{\theta_{k+1}}^{k+1}(X_{\theta_{k+1}}^{k,x}, \alpha_{\theta_{k+1}}^k, e_{k+1}), \theta^{(k)}, \theta_{k+1}, e^{(k)}, e_{k+1}, \alpha^\varepsilon\big) \\
& \quad \eta_1(de_{k+1}) d\theta_{k+1} \Big| \mathcal{F}_{\theta_k} \Big] \\
\geq\;& \mathbb{E}\Big[\int_{\theta_k}^T f_t^k(X_t^{k,x}, \alpha_t^k, \theta^{(k)}, e^{(k)}) \gamma_t^k(\theta^{(k)}, e^{(k)}) dt \\
& + G_T^k(X_T^{k,x}, \theta^{(k)}, e^{(k)}) \gamma_T^k(\theta^{(k)}, e^{(k)}) \\
& + \int_{\theta_k}^T \int_E V_{k+1}\big(\Gamma_{\theta_{k+1}}^{k+1}(X_{\theta_{k+1}}^{k,x}, \alpha_{\theta_{k+1}}^k, e_{k+1}), \theta^{(k)}, \theta_{k+1}, e^{(k)}, e_{k+1}\big) \\
& \quad \eta_1(de_{k+1}) d\theta_{k+1} \Big| \mathcal{F}_{\theta_k} \Big] - \varepsilon.
\end{aligned}
$$

From the arbitrariness of $\alpha^k \in \mathcal{A}_\mathbb{F}^k$ and $\varepsilon > 0$, we obtain the required inequality: $V_k(x, \theta^{(k)}, e^{(k)}) \geq \bar{V}_k(x, \theta^{(k)}, e^{(k)})$, and the proof is complete. $\qquad\square$

5 The case of enlarged filtration with multiple random times

In this section, we consider the case where the random times are not assumed to be ordered. In other words, this means that one has access to the default times themselves with their indexes, and not only to the ranked default times. This general case can actually be derived from the case of successive random times associated with suitable auxiliary marks. Let us consider the progressive enlargement of filtration from $\mathbb{F}$ to $\mathbb{G}$ with multiple random times $(\tau_1, \ldots, \tau_n)$ associated with the marks $(\zeta_1, \ldots, \zeta_n)$. Denote by $\hat{\tau}_1 \leq \ldots \leq \hat{\tau}_n$ the corresponding ranked times, and by ι_i the index mark (valued in $\{1, \ldots, n\}$) of the i-th order statistic of $(\tau_1, \ldots, \tau_n)$ for $i = 1, \ldots, n$. Then, it is clear that the progressive enlargement of filtration of $\mathbb{F}$ with the successive random times $(\hat{\tau}_1, \ldots, \hat{\tau}_n)$ together with the marks $(\iota_1, \zeta_{\iota_1}, \ldots, \iota_n, \zeta_{\iota_n})$ leads to the filtration $\mathbb{G}$, so that one can apply the results of the previous sections. For simplicity of notations, we shall focus on the case of two random times τ_1 and τ_2, associated to the marks ζ_1 and ζ_2 valued in E Borel space of $\mathbb{R}^m$.

The decomposition of optional and predictable process with respect to this progressive enlargement of filtration is given by the following lemma, which is derived from Lemma 2.1, with the specific feature that we have also to take into account the index of the order statistic in (τ_1, τ_2).

Lemma 5.1 *Any* $\mathbb{G}$*-optional (resp. predictable) process* $Y = (Y_t)_{t \geq 0}$ *is represented as*

$$
\begin{aligned}
Y_t \quad = \quad & Y_t^0 1_{t<\hat{\tau}_1} + Y_t^{1,1}(\tau_1, \zeta_1)1_{\tau_1 \leq t < \tau_2} + Y_t^{1,2}(\tau_2, \zeta_2)1_{\tau_2 \leq t < \tau_1} + Y_t^2(\tau_1, \tau_2, \zeta_1, \zeta_2)1_{t \geq \hat{\tau}_2}, \\
(resp. = \quad & Y_t^0 1_{t \leq \hat{\tau}_1} + Y_t^{1,1}(\tau_1, \zeta_1)1_{\tau_1 < t \leq \tau_2} + Y_t^{1,2}(\tau_2, \zeta_2)1_{\tau_2 < t \leq \tau_1} + Y_t^2(\tau_1, \tau_2, \zeta_1, \zeta_2)1_{t > \hat{\tau}_2}),
\end{aligned}
$$

for all $t \geq 0$, *where* $Y^0 \in \mathcal{O}_\mathbb{F}$ *(resp.* $\mathcal{P}_\mathbb{F}$*),* $Y^{1,1}, Y^{1,2} \in \mathcal{O}_\mathbb{F}^1(\mathbb{R}_+, E)$ *(resp.* $\mathcal{P}_\mathbb{F}^1(\mathbb{R}_+, E)$*), and* $Y^2 \in \mathcal{O}_\mathbb{F}^2(\mathbb{R}_+^2, E^2)$ *(resp.* $\mathcal{P}_\mathbb{F}^2(\mathbb{R}_+^2, E^2)$*).*

Any $Y \in \mathcal{O}_\mathbb{G}$ (resp. $\mathcal{P}_\mathbb{G}$) can then be identified with a quadruple $(Y^0, Y^{1,1}, Y^{1,2}, Y^2)$ $\in \mathcal{O}_\mathbb{F} \times \mathcal{O}_\mathbb{F}^1(\mathbb{R}_+, E) \times \mathcal{O}_\mathbb{F}^1(\mathbb{R}_+, E) \times \mathcal{O}_\mathbb{F}^2(\mathbb{R}_+^2, E^2)$ (resp. $\mathcal{P}_\mathbb{F} \times \mathcal{P}_\mathbb{F}^1(\mathbb{R}_+, E) \times \mathcal{P}_\mathbb{F}^1(\mathbb{R}_+, E) \times \mathcal{P}_\mathbb{F}^2(\mathbb{R}_+^2, E^2)$).

Similarly as in Section 1, we now make a density hypothesis on the conditional distribution of $(\tau_1, \tau_2, \zeta_1, \zeta_2)$ given the reference information. We assume that there exists a $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+^2) \otimes \mathcal{B}(E^2)$-measurable map $(t, \omega, \theta_1, \theta_2, e_1, e_2) \to \gamma_t(\omega, \theta_1, \theta_2, e_1, e_2)$ such that

(DH) $\quad \mathbb{P}\big[(\tau_1, \tau_2, \zeta_1, \zeta_2) \in d\theta de | \mathcal{F}_t\big] \quad = \quad \gamma_t(\theta_1, \theta_2, e_1, e_2)d\theta_1 d\theta_2 \eta(de_1)\eta(de_2), \quad a.s.$

where η is a nonnegative measure on $\mathcal{B}(E)$.

We next introduce some useful notations. We denote by γ_0 the $\mathbb{F}$-optional process defined by

$$
\gamma_t^0 \quad = \quad \mathbb{P}[\tau_1 > t | \mathcal{F}_t] \quad = \quad \int_{E^2} \int_{[t,\infty)^2} \gamma_t(\theta_1, \theta_2, e_1, e_2)d\theta_1 d\theta_2 \eta(de_1)\eta(de_2),
$$

and we denote by $(t, \omega, \theta_1, e_1) \to \gamma_t^{1,1}(\theta_1, e_1)$, and $(t, \omega, \theta_2, e_2) \to \gamma_t^{1,2}(\theta_2, e_2)$, $t \geq 0$, the $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(E)$-measurable maps defined by

$$
\gamma_t^{1,1}(\theta_1, e_1) \quad = \quad \int_E \int_t^\infty \gamma_t(\theta_1, \theta_2, e_1, e_2)d\theta_2 \eta(de_2),
$$
$$
\gamma_t^{1,2}(\theta_2, e_2) \quad = \quad \int_E \int_t^\infty \gamma_t(\theta_1, \theta_2, e_1, e_2)d\theta_1 \eta(de_1),
$$

so that

$$
\mathbb{P}[\tau_2 > t | \mathcal{F}_t] = \int_E \int_0^\infty \gamma_t^{1,1}(\theta_1, e_1)d\theta_1 \eta(de_1), \qquad \mathbb{P}[\tau_1 > t | \mathcal{F}_t] = \int_E \int_0^\infty \gamma_t^{1,2}(\theta_2, e_2)d\theta_2 \eta(de_2).
$$

The next result, which is analog to Proposition 2.1, provides a backward induction formula involving $\mathbb{F}$-expectations for the computation of expectation functionals of $\mathbb{G}$-optional processes.

Proposition 5.1 *Let* $Y = (Y^0, Y^{1,1}, Y^{1,2}, Y^2)$ *and* $Z = (Z^0, Z^{1,1}, Z^{1,2}, Z^2)$ *be two nonnegative (or bounded)* $\mathbb{G}$*-optional processes, and fix* $T \in (0, \infty)$.
The expectation $\mathbb{E}[\int_0^T Y_t dt + Z_T]$ *can be computed in a backward induction as*

$$
\mathbb{E}\Big[\int_0^T Y_t dt + Z_T\Big] \quad = \quad J_0
$$

where the $(J_0, J_{1,1}, J_{1,2}, J_2)$ are given by

$$J_2(\theta_1, \theta_2, e_1, e_2) = \mathbb{E}\Big[\int_{\theta_1 \vee \theta_2}^{T} Y_t^2 \gamma_t(\theta_1, \theta_2, e_1, e_2)dt + Z_T^2 \gamma_T(\theta_1, \theta_2, e_1, e_2) \Big| \mathcal{F}_{\theta_1 \vee \theta_2} \Big]$$

$$J_{1,1}(\theta_1, e_1) = \mathbb{E}\Big[\int_{\theta_1}^{T} Y_t^{1,1} \gamma_t^{1,1}(\theta_1, e_1)dt + Z_T^{1,1} \gamma_T^{1,1}(\theta_1, e_1)$$

$$+ \int_E \int_{\theta_1}^{T} J_2(\theta_1, \theta_2, e_1, e_2) d\theta_2 \eta(de_2) \Big| \mathcal{F}_{\theta_1} \Big]$$

$$J_{1,2}(\theta_2, e_2) = \mathbb{E}\Big[\int_{\theta_2}^{T} Y_t^{1,2} \gamma_t^{1,2}(\theta_2, e_2)dt + Z_T^{1,2} \gamma_T^{1,2}(\theta_2, e_2)$$

$$+ \int_E \int_{\theta_1}^{T} J_2(\theta_1, \theta_2, e_1, e_2) d\theta_2 \eta(de_2) \Big| \mathcal{F}_{\theta_2} \Big]$$

$$J_0 = \mathbb{E}\Big[\int_0^{T} Y_t^0 \gamma_t^0 dt + Z_T^0 \gamma_T^0$$

$$+ \int_E \int_0^{T} J_{1,1}(\theta_1, e_1) d\theta_1 \eta(de_1) + \int_E \int_0^{T} J_{1,2}(\theta_2, e_2) d\theta_2 \eta(de_2) \Big].$$

Let us now formulate the general stochastic control problem in this framework. A control is a $\mathbb{G}$-predictable process $\alpha = (\alpha^0, \alpha^{1,1}, \alpha^{1,2}, \alpha^2) \in \mathcal{P}_{\mathbb{F}} \times \mathcal{P}_{\mathbb{F}}^1(\mathbb{R}_+, E) \times \mathcal{P}_{\mathbb{F}}^1(\mathbb{R}_+, E) \times \mathcal{P}_{\mathbb{F}}^2(\mathbb{R}_+^2, E^2)$, where α^0, $\alpha^{1,1}$, $\alpha^{1,2}$ and α^2 are valued respectively in A_0, $A_{1,1}$, $A_{1,2}$ and A_2, Borel sets of some Euclidian space. We denote by $A = A_0 \times A_{1,1} \times A_{1,2} \times A_2$, and by $\mathcal{A}_{\mathbb{G}}$ the set of admissible control processes, which is a product space $\mathcal{A}_{\mathbb{F}}^0 \times \mathcal{A}_{\mathbb{F}}^{1,1} \times \mathcal{A}_{\mathbb{F}}^{1,2} \times \mathcal{A}_{\mathbb{F}}^2$, where $\mathcal{A}_{\mathbb{F}}^0$, $\mathcal{A}_{\mathbb{F}}^{1,1}$, $\mathcal{A}_{\mathbb{F}}^{1,2}$ and $\mathcal{A}_{\mathbb{F}}^2$ are some separable metric spaces respectively in $\mathcal{P}_{\mathbb{F}}(A_0)$, $\mathcal{P}_{\mathbb{F}}^1(\mathbb{R}_+, E; A_{1,1})$, $\mathcal{P}_{\mathbb{F}}^1(\mathbb{R}_+, E; A_{1,2})$ and $\mathcal{P}_{\mathbb{F}}^2(\mathbb{R}_+^2, E^2; A_2)$.

We are next given a collection of measurable mappings:

$$(x, \alpha^0) \in \mathbb{R}^d \times \mathcal{A}_{\mathbb{F}}^0 \longmapsto X^{0,x,\alpha^0} \in \mathcal{O}_{\mathbb{F}}$$
$$(x, \alpha^{1,1}) \in \mathbb{R}^d \times \mathcal{A}_{\mathbb{F}}^{1,1} \longmapsto X^{1,1,x,\alpha^{1,1}} \in \mathcal{O}_{\mathbb{F}}^1(\mathbb{R}_+, E)$$
$$(x, \alpha^{1,2}) \in \mathbb{R}^d \times \mathcal{A}_{\mathbb{F}}^{1,2} \longmapsto X^{1,2,x,\alpha^{1,2}} \in \mathcal{O}_{\mathbb{F}}^1(\mathbb{R}_+, E)$$
$$(x, \alpha^2) \in \mathbb{R}^d \times \mathcal{A}_{\mathbb{F}}^2 \longmapsto X^{2,x,\alpha^2} \in \mathcal{O}_{\mathbb{F}}^2(\mathbb{R}_+^2, E^2),$$

such that we have the initial data

$$X_0^{0,x,\alpha^0} = x, \quad \forall x \in \mathbb{R}^d,$$
$$X_{\theta_1}^{1,1,\xi,\alpha^{1,1}}(\theta_1, e_1) = \xi, \quad \forall \xi \ \mathcal{F}_{\theta_1} - \text{measurable},$$
$$X_{\theta_2}^{1,2,\xi,\alpha^{1,2}}(\theta_2, e_2) = \xi, \quad \forall \xi \ \mathcal{F}_{\theta_2} - \text{measurable},$$
$$X_{\theta_1 \vee \theta_2}^{2,\xi,\alpha^2}(\theta_1, \theta_2, e_1, e_2) = \xi, \quad \forall \xi \ \mathcal{F}_{\theta_1 \vee \theta_2} - \text{measurable}.$$

We are also given a collection of maps $\Gamma^{1,1}$, $\Gamma^{1,2}$, on $\mathbb{R}_+ \times \Omega \times \mathbb{R}^d \times A_0 \times E$, $\Gamma^{2,1}$ on $\mathbb{R}_+ \times \Omega \times \mathbb{R}^d \times A_{1,1} \times E$ and $\Gamma^{2,2}$ on $\mathbb{R}_+ \times \Omega \times \mathbb{R}^d \times A_{1,2} \times E$ such that

$$(t, \omega, x, a, e) \mapsto \Gamma_t^{1,1}(\omega, x, a, e), \ \Gamma_t^{1,2}(\omega, x, a, e)$$
$$\text{are} \quad \mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(A_0) \otimes \mathcal{B}(E) - \text{measurable}$$
$$(t, \omega, x, a, e) \mapsto \Gamma_t^{2,1}(\omega, x, a, e) \ \text{is} \ \mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(A_{1,1}) \otimes \mathcal{B}(E) - \text{measurable}$$
$$(t, \omega, x, a, e) \mapsto \Gamma_t^{2,2}(\omega, x, a, e) \ \text{is} \ \mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(A_{1,2}) \otimes \mathcal{B}(E) - \text{measurable}$$

18

The controlled state process is then given by the mapping

$$(x, \alpha) \in \mathbb{R}^d \times \mathcal{A}_\mathbb{G} \quad \longmapsto \quad X^{x,\alpha} \in \mathcal{O}_\mathbb{G},$$

where for $\alpha = (\alpha^0, \alpha^{1,1}, \alpha^{1,2}, \alpha^2)$, $X^{x,\alpha}$ is the process equal to

$$X_t^{x,\alpha} = \bar{X}_t^0 1_{t<\hat{\tau}_1} + \bar{X}_t^{1,1}(\tau_1, \zeta_1) 1_{\tau_1 \leq t < \tau_2} + \bar{X}_t^{1,2}(\tau_2, \zeta_2) 1_{\tau_2 \leq t < \tau_1} + \bar{X}_t^2(\tau_1, \tau_2, \zeta_1, \zeta_2) 1_{t \geq \hat{\tau}_2},$$

with $(\bar{X}^0, \bar{X}^{1,1}, \bar{X}^{1,2}, \bar{X}^2) \in \mathcal{O}_\mathbb{F} \times \mathcal{O}_\mathbb{F}^1(\mathbb{R}_+, E) \times \mathcal{O}_\mathbb{F}^1(\mathbb{R}_+, E) \times \mathcal{O}_\mathbb{F}^2(\mathbb{R}_+^2, E^2)$ given by

$$\begin{aligned}
\bar{X}_0 &= X^{0,x,\alpha^0} \\
\bar{X}^{1,1}(\theta_1, e_1) &= X^{1,1,\Gamma_{\theta_1}^{1,1}(\bar{X}_{\theta_1}^0, \alpha_{\theta_1}^0, e_1), \alpha^{1,1}}(\theta_1, e_1) \\
\bar{X}^{1,2}(\theta_2, e_2) &= X^{1,2,\Gamma_{\theta_2}^{1,2}(\bar{X}_{\theta_2}^0, \alpha_{\theta_2}^0, e_2), \alpha^{1,2}}(\theta_2, e_2) \\
\bar{X}^2(\theta_1, \theta_2, e_1, e_2) &= \begin{cases} X^{2,\Gamma_{\theta_2}^{2,2}(\bar{X}_{\theta_2}^{1,1}, \alpha_{\theta_2}^{1,1}, e_2), \alpha^2}(\theta_1, \theta_2, e_1, e_2) & \text{if } \theta_1 \leq \theta_2 \\ X^{2,\Gamma_{\theta_1}^{2,1}(\bar{X}_{\theta_1}^{1,2}, \alpha_{\theta_1}^{1,2}, e_1), \alpha^2}(\theta_1, \theta_2, e_1, e_2) & \text{if } \theta_2 < \theta_1. \end{cases}
\end{aligned}$$

The interpretation is the following: X^0 is the controlled state process before any default, $X^{1,1}$ (resp. $X^{1,2}$) is the controlled state process between τ_1 and τ_2 (resp. between τ_2 and τ_1) if the default of index 1 (resp. index 2) occurs first, and X^2 is the controlled state process after both defaults. Moreover, $\Gamma^{1,1}$ (resp. $\Gamma^{1,2}$) represents the jump of X^0 at τ_1 (resp. τ_2) if the default of index 1 (resp. index 2) occurs first, and $\Gamma^{2,2}$ (resp. $\Gamma^{2,1}$) represents the jump of $X^{1,1}$ (resp. $X^{1,2}$) at τ_2 (resp. τ_1) when the default of index 2 (resp. index 1) occurs in second after index 1 (resp. index 2).

For a fixed finite horizon $T < \infty$, we are given a nonnegative map G_T on $\Omega \times \mathbb{R}^d$ such that $(\omega, x) \mapsto G_T(\omega, x)$ is $\mathcal{G}_T \otimes \mathcal{B}(\mathbb{R}^d)$-measurable, thus in the form

$$\begin{aligned}
G_T(x) &= G_T^0(x) 1_{T<\hat{\tau}_1} + G_T^{1,1}(x, \tau_1, \zeta_1) 1_{\tau_1 \leq T < \tau_2} + G_T^{1,2}(x, \tau_2, \zeta_2) 1_{\tau_2 \leq T < \tau_1} \\
&\quad + G_T^2(x, \tau_1, \tau_2, \zeta_1, \zeta_2) 1_{\hat{\tau}_2 \leq T},
\end{aligned}$$

where G_T^0 is $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}^d)$-measurable, $G_T^{1,1}$, $G_T^{1,2}$ are $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(E)$-measurable, and G_T^2 is $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}_+^2) \otimes \mathcal{B}(E^2)$-measurable. The running gain function is given by a nonnegative map f on $\Omega \times \mathbb{R}^d \times A$ such that $(t, \omega, x, a) \mapsto f_t(\omega, x, a)$ is $\mathcal{O}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(A)$-measurable, and which may be decomposed as

$$\begin{aligned}
f_t(x, a) &= f_t^0(x, a_0) 1_{t<\hat{\tau}_1} + f_t^{1,1}(x, a_{1,1}, \tau_1, \zeta_1) 1_{\tau_1 \leq t < \tau_2} + f_t^{1,2}(x, a_{1,2}, \tau_2, \zeta_2) 1_{\tau_2 \leq t < \tau_1} \\
&\quad + f_t^2(x, a_2, \tau_1, \tau_2, \zeta_1, \zeta_2) 1_{\hat{\tau}_2 \leq T},
\end{aligned}$$

for $a = (a_0, a_{1,1}, a_{1,2}, a_2) \in A = A_0 \times A_{1,1} \times A_{1,2} \times A_2$, where f^0 is $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(A_0)$-measurable, and $f^{1,1}$ is $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(A_{1,1}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(E)$-measurable, $f^{1,2}$ is $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(A_{1,2}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(E)$-measurable and f^2 is $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(A_2) \otimes \mathcal{B}(\mathbb{R}_+^2) \otimes \mathcal{B}(E^2)$-measurable.

The value function for the stochastic control problem is then defined by

$$V_0(x) = \sup_{\alpha \in \mathcal{A}_\mathbb{G}} \mathbb{E}\left[\int_0^T f_t(X_t^{x,\alpha}, \alpha_t) dt + G_T(X_T^{x,\alpha}) \right], \quad x \in \mathbb{R}^d.$$

19

The main result of this section provides a decomposition of the value function in the reference filtration, which is analog to the decomposition in Theorem 4.1. To alleviate the notations, we omit the dependence of the state process in the controls and in the parameters θ, e, when there is no ambiguity.

Theorem 5.1 *The value function V_0 is obtained from the backward induction formula*

$$
\begin{aligned}
V_2(x, \theta_1, \theta_2, e_1, e_2) &= \operatorname*{ess\,sup}_{\alpha^2 \in \mathcal{A}_{\mathbb{F}}^2} \mathbb{E}\Big[\int_{\theta_1 \vee \theta_2}^T f_t^2(X_t^{2,x}, \alpha_t^2, \theta_1, \theta_2, e_1, e_2)\gamma_t(\theta_1, \theta_2, e_1, e_2)dt \\
&\qquad + G_T^2(X_T^{2,x}, \theta_1, \theta_2, e_1, e_2)\gamma_T(\theta_1, \theta_2, e_1, e_2)\Big|\mathcal{F}_{\theta_1 \vee \theta_2}\Big] \\
V_{1,1}(x, \theta_1, e_1) &= \operatorname*{ess\,sup}_{\alpha^{1,1} \in \mathcal{A}_{\mathbb{F}}^{1,1}} \mathbb{E}\Big[\int_{\theta_1}^T f_t^{1,1}(X_t^{1,1,x}, \alpha_t^{1,1}, \theta_1, e_1)\gamma_t^{1,1}(\theta_1, e_1)dt \\
&\qquad + G_T^{1,1}(X_T^{1,1,x}, \theta_1, e_1)\gamma_T^{1,1}(\theta_1, e_1) \\
&\qquad + \int_{\theta_1}^T \int_E V_2\big(\Gamma_{\theta_2}^{2,2}(X_{\theta_2}^{1,1,x}, \alpha_{\theta_2}^{1,1}, e_2), \theta_1, \theta_2, e_1, e_2\big)\eta(de_2)d\theta_2\Big|\mathcal{F}_{\theta_1}\Big] \\
V_{1,2}(x, \theta_2, e_2) &= \operatorname*{ess\,sup}_{\alpha^{1,2} \in \mathcal{A}_{\mathbb{F}}^{1,2}} \mathbb{E}\Big[\int_{\theta_2}^T f_t^{1,2}(X_t^{1,2,x}, \alpha_t^{1,2}, \theta_2, e_2)\gamma_t^{1,2}(\theta_2, e_2)dt \\
&\qquad + G_T^{1,2}(X_T^{1,2,x}, \theta_2, e_2)\gamma_T^{1,2}(\theta_2, e_2) \\
&\qquad + \int_{\theta_1}^T \int_E V_2\big(\Gamma_{\theta_1}^{2,1}(X_{\theta_1}^{1,2,x}, \alpha_{\theta_1}^{1,2}, e_1), \theta_1, \theta_2, e_1, e_2\big)\eta(de_1)d\theta_1\Big|\mathcal{F}_{\theta_2}\Big] \\
V_0(x) &= \sup_{\alpha^0 \in \mathcal{A}_{\mathbb{F}}^0} \mathbb{E}\Big[\int_0^T f_t^0(X_t^{0,x}, \alpha_t^0)\gamma_t^0 dt + G_T^0(X_T^{0,x})\gamma_T^0 \\
&\qquad + \int_0^T \int_E V_{1,1}\big(\Gamma_{\theta_1}^{1,1}(X_{\theta_1}^{0,x}, \alpha_{\theta_1}^0, e_1), \theta_1, e_1\big)\eta(de_1)d\theta_1 \\
&\qquad + \int_0^T \int_E V_{1,2}\big(\Gamma_{\theta_2}^{1,2}(X_{\theta_2}^{0,x}, \alpha_{\theta_2}^0, e_2), \theta_2, e_2\big)\eta(de_2)d\theta_2\Big],
\end{aligned}
$$

for all $(\theta_1, \theta_2) \in [0,T]^2$, $(e_1, e_2) \in E^2$.

Remark 5.1 As mentioned in Remark 4.1, the value functions V_2, $V_{1,1}$ and $V_{1,2}$ correspond to standard stochastic control problem in the $\mathbb{F}$-filtration. This is also the case for V_0 in the decomposition formula of Theorem 5.1. Indeed, denote by V_1 the map on $\Omega \times [0,T] \times \mathbb{R}^d \times A_0$:

$$
V_1(x, \theta, a_0) = \int_E V_{1,1}(\Gamma_\theta^{1,1}(x, a_0, e), \theta, e) + V_{1,2}(\Gamma_\theta^{1,2}(x, a_0, e), \theta, e)\, \eta(de).
$$

Then, V_0 is computed from the stochastic control problem in the $\mathbb{F}$-filtration with the terminal gain function G_T^0 weighted by the $\mathcal{F}_T$-measurable random variable γ_T^0, and with the running gain functions $f^0\gamma^0$ and V_1:

$$
V_0(x) = \sup_{\alpha^0 \in \mathcal{A}_{\mathbb{F}}} \mathbb{E}\Big[G_T^0(X_T^{0,x})\gamma_T^0 + \int_0^T f_t^0(X_t^{0,x}, \alpha_t^0)\gamma_t^0 + V_1(X_t^{0,x}, t, \alpha_t^0)dt\Big].
$$

6 Applications in mathematical finance

6.1 Indifference pricing of defaultable claims

We consider a stock subject to a single counterparty default at a random time τ, which induces a jump of random relative size ζ valued in $E \subset (-1, \infty)$. The price process of the stock is described by

$$S_t = S_t^0 1_{t < \tau} + S_t^1(\tau, \zeta) 1_{t \geq \tau},$$

where S^0 is governed by

$$dS_t^0 = S_t^0 (b_t^0 dt + \sigma_t^0 dW_t),$$

and the indexed process $S^1(\theta, e)$, $(\theta, e) \in \mathbb{R}_+ \times E$ is given by

$$
\begin{aligned}
dS_t^1(\theta, e) &= S_t^1(\theta, e)\big(b_t^1(\theta, e)dt + \sigma_t^1(\theta, e)dW_t\big), \quad t \geq \theta, \\
S_\theta^1(\theta, e) &= S_\theta^0.(1 + e).
\end{aligned}
$$

Here W is a $(\mathbb{P}, \mathbb{F})$-Brownian motion, b^0, $\sigma^0 > 0$ are $\mathbb{F}$-adapted processes, b^1, $\sigma^1 > 0 \in \mathcal{O}_{\mathbb{F}}^1(\mathbb{R}_+, E)$. The market information is represented by the progressive enlarged filtration $\mathbb{G} = \mathbb{F} \vee \mathbb{D}$, with $\mathbb{D} = (\mathcal{D}_t)_{t \geq 0}$, $\mathcal{D}_t = \cap_{\varepsilon > 0}\{\sigma(\zeta 1_{\tau \leq s}, s \leq t + \varepsilon) \vee \sigma(1_{\tau \leq s}, s \leq t + \varepsilon)\}$. An investor can trade in a riskless bond with zero interest rate, and in the defaultable stock. Her trading strategy is a $\mathbb{G}$-predictable process $\alpha = (\alpha^0, \alpha^1) \in \mathcal{P}_{\mathbb{F}} \times \mathcal{P}_{\mathbb{F}}^1(\mathbb{R}_+, E)$ representing the amount traded in the stock. We allow constraints on trading strategy by considering closed sets A_0 and A_1 in which the controls α^0 and α^1 take values. Notice also that A_0 and A_1 may differ. The controlled wealth process of the investor is then given by

$$X_t = X_t^0 1_{t < \tau} + X_t^1(\tau, \zeta) 1_{t \geq \tau}, \tag{6.1}$$

where X^0 is the wealth process before the default, and governed by

$$dX_t^0 = \alpha_t^0 \frac{dS_t^0}{S_t^0} = \alpha_t^0 (b_t^0 dt + \sigma_t^0 dW_t),$$

and $X^1(\theta, e)$ is the wealth indexed process after-default, governed by

$$
\begin{aligned}
dX_t^1(\theta, e) &= \alpha_t^1(\theta, e)\frac{dS_t^1(\theta, e)}{S_t^1(\theta, e)} = \alpha_t^1(\theta, e)\big(b_t^1(\theta, e)dt + \sigma_t^1(\theta, e)dW_t\big), \quad t \geq \theta \\
X_\theta^1(\theta, e) &= X_\theta^0 + \alpha_\theta^0 e.
\end{aligned}
$$

Let us now consider a defaultable contingent claim with payoff at maturity T given by

$$H_T = H_T^0 1_{T < \tau} + H_T^1(\tau, \zeta) 1_{\tau \leq T},$$

where H_T^0 is a bounded $\mathcal{F}_T$-measurable random variable, and $H_T^1(,)$ is a bounded $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(E)$-measurable map. We use the popular indifference pricing criterion for valuing this defaultable claim. We are then given an exponential utility function U on $\mathbb{R}$, i.e.

$$U(x) = -\exp(-px), \quad x \in \mathbb{R},$$

for some $p > 0$, and we consider the optimal investment problem for an agent delivering the defaultable claim at maturity T:

$$V_0^H(x) \;=\; \sup_{\alpha \in \mathcal{A}_{\mathbb{G}}} \mathbb{E}\big[U(X_T^{x,\alpha} - H_T)\big]. \tag{6.2}$$

Here $X^{x,\alpha}$ is the wealth process in (6.1) controlled by the trading strategy α, and starting from x at time 0. We denote by V_0 the value function for the optimal investment problem without the defaultable claim, i.e. when $H_T = 0$ in (6.2), and the indifference price for H_T is the amount of initial capital such that the investor is indifferent between holding or not the defaultable claim. It is then defined as the unique number π such that

$$V_0^H(x + \pi) \;=\; V_0(x).$$

A similar problem (without unpredictable mark ζ) was recently considered in [15] and [1] by using a global $\mathbb{G}$-filtration approach under **(H)** hypothesis, see also [20]. The paper [14] studied an optimal investment problem with power utility functions under a single counterparty default by using a density approach for decomposing the problem in the $\mathbb{F}$-filtration. We follow this methodology and solve the stochastic control problem (6.2) by applying the $\mathbb{F}$-decomposition method. From Theorem 4.1, the value function V_0^H is obtained in two steps via the resolution of the after-default problem

$$V_1^H(x, \theta, e) \;=\; \operatorname*{ess\,sup}_{\alpha^1 \in \mathcal{A}_{\mathbb{F}}^1} \mathbb{E}\Big[U\big(X_T^{1,x}(\theta, e) - H_T^1(\theta, e)\big)\gamma_T(\theta, e)\Big|\mathcal{F}_\theta\Big], \tag{6.3}$$

and then via the resolution of the before-default problem

$$V_0^H(x) \;=\; \sup_{\alpha^0 \in \mathcal{A}_{\mathbb{F}}^0} \mathbb{E}\Big[U(X_T^{0,x} - H_T^0)\gamma_T^0 + \int_0^T \int_E V_1^H(X_\theta^{0,x} + \alpha_\theta^0 e, \theta, e)\eta(de)d\theta\Big]. \tag{6.4}$$

• *Solution to the after-default problem.*
For fixed $(\theta, e) \in [0, T] \times E$, problem (6.3) is a classical utility maximization problem with random endowment in the complete market model after default described by the indexed price process $S^1(\theta, e)$. Indeed, notice that we can remove the positive term $\gamma_T(\theta, e)$ in (6.3) by defining the "modified claim" $\tilde{H}_T^1(\theta, e) = H_T^1(\theta, e) + \frac{1}{p}\ln\gamma_T(\theta, e)$ so that

$$V_1^H(x, \theta, e) \;=\; \operatorname*{ess\,sup}_{\alpha^1 \in \mathcal{A}_{\mathbb{F}}^1} \mathbb{E}\Big[U\big(X_T^{1,x}(\theta, e) - \tilde{H}_T^1(\theta, e)\big)\Big|\mathcal{F}_\theta\Big]. \tag{6.5}$$

This problem was addressed by several methods in the literature, and we know from dynamic programming and BSDE methods (see [22] or [8])) that

$$V_1^H(x, \theta, e) \;=\; U\big(x - Y_\theta^{1,H}(\theta, e)\big)$$

where $Y^{1,H}(\theta, e)$ is the unique bounded solution to the BSDE

$$Y_t^{1,H}(\theta, e) \;=\; H_T^1(\theta, e) + \frac{1}{p}\ln\gamma_T(\theta, e) + \int_t^T f^1(r, Z_r^{1,H}, \theta, e)dr - \int_t^T Z_r^{1,H}dW_r$$

22

and the generator f^1 is the $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(E)$-measurable map defined by

$$f^1(t, z, \theta, e) = -\frac{b_t^1(\theta, e)}{\sigma_t^1(\theta, e)} z - \frac{1}{2p} \left(\frac{b_t^1(\theta, e)}{\sigma_t^1(\theta, e)} \right)^2 + \frac{p}{2} \inf_{a \in A_1} \left| \left(z + \frac{1}{p} \frac{b_t^1(\theta, e)}{\sigma_t^1(\theta, e)} \right) - a\sigma_t^1(\theta, e) \right|^2.$$

● *Global solution*

The global solution is finally obtained from the resolution of the before-default problem, which is then reduced to

$$V_0^H(x) = \sup_{\alpha^0 \in \mathcal{A}_{\mathbb{F}}^0} \mathbb{E} \left[U(X_T^{0,x} - H_T^0) \gamma_T^0 + \int_0^T \int_E U(X_\theta^{0,x} + \alpha_\theta^0 e - Y_\theta^{1,H}(e)) \eta(de) d\theta \right].$$

From the additive dependence of the wealth process $X^{0,x}$ in function of x, and the exponential form of the utility function U, we know that the value function V_0^H is in the form

$$V_0^H(x) = U(x - Y_0^{0,H}),$$

for some quantity $Y_0^{0,H}$ independent of x, and which may be characterized by dynamic programming methods in the $\mathbb{F}$-filtration. This can be achieved either via PDE methods in a Markovian setting, or via BSDE methods in the general case. The BSDE associated to $Y^{0,H}$ is

$$Y_t^{0,H} = H_T^0 + \frac{1}{p} \ln \gamma_T^0 + \int_t^T f^{0,H}(r, Y_r^{0,H}, Z_r^{0,H}) dr - \int_t^T Z_r^{0,H} dW_r, \qquad (6.6)$$

where the generator $f^{0,H}$ is the $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$-measurable map defined by

$$f^{0,H}(t, y, z) = -\frac{b_t^0}{\sigma_t^0} z - \frac{1}{2p} \left(\frac{b_t^0}{\sigma_t^0} \right)^2 \qquad (6.7)$$
$$+ \frac{p}{2} \inf_{a \in A_0} \left| \left(z + \frac{1}{p} \frac{b_t^0}{\sigma_t^0} \right) - a\sigma_t^0 + \frac{2}{p} U(y) \int_E U(ae - Y_t^{1,H}(t, e)) \eta(de) \right|^2.$$

The solution to the optimal investment problem without defaultable claim is obtained similarly as for the case with claim, by considering $H = 0$. We thus have $V_0(x) = R(x - Y_0^0)$, where the BSDE associated to Y^0 is given by

$$Y_t^0 = \frac{1}{p} \ln \gamma_T^0 + \int_t^T f^0(r, Y_r^0, Z_r^0) dr - \int_t^T Z_r^0 dW_r,$$

with a generator f^0 as in (6.7) for $H = 0$, i.e. $Y^{1,H}$ replaced by Y^1 solution to the BSDE

$$Y_t^1(\theta, e) = \frac{1}{p} \ln \gamma_T(\theta, e) + \int_t^T f^1(r, Z_r^1, \theta, e) dr - \int_t^T Z_r^1 dW_r.$$

Finally, the indifference price is given by

$$\pi = Y_0^{0,H} - Y_0^0.$$

Remark 6.1 Notice that the quadratic generator $f^{0,H}$ in (6.7) of the BSDE (6.6) is not standard due to the additional term arising from the integral gain involving $Y^{1,H}$. However, one can prove existence and uniqueness of this BSDE and obtain a verification theorem relating the solution of this BSDE to the original value function by choosing a suitable set of admissible controls $\mathcal{A}_{\mathbb{G}} = \mathcal{A}_{\mathbb{F}}^0 \times \mathcal{A}_{\mathbb{F}}^1$. The details are provided in the companion paper [13]. Actually, in this related paper, we consider a multi-dimensional extension of the above model with assets subject to successive counterparty default times, and we apply the $\mathbb{F}$-decomposition method for solving the indifference pricing of defaultable claims, including credit derivatives such as k-th default swap.

6.2 Optimal investment under bilateral counterparty risk

We consider a portfolio with two names, each one subject to an external counterparty default, but also to the default of the other one due to a contagion effect. We denote by S^1 and S^2 the value process of these two names, by τ_1 and τ_2 their default times, not necessarily ordered, and by $\hat{\tau}_1 = \min(\tau_1, \tau_2)$, $\hat{\tau}_2 = \max(\tau_1, \tau_2)$. Once the name i defaults at random time τ_i, meaning that the value of S^i drops to zero, it also incurs a jump (drop or gain) on the other value process S^j, $i, j \in \{1, 2\}$, $i \neq j$.

The reference filtration $\mathbb{F}$ is the filtration generated by a two-dimensional Brownian motion $W = (W^1, W^2)$, driving the evolution of the names in absence of defaults, and the global market information is represented by $\mathbb{G} = \mathbb{F} \vee \mathbb{D}^1 \vee \mathbb{D}^2$, with $\mathbb{D}^i = (\mathcal{D}_t^i)_{t \geq 0}$, $\mathcal{D}_t^i = \cap_{\varepsilon > 0} \sigma(1_{\tau_i \leq s}, s \leq t + \varepsilon)$, $i = 1, 2$.

The $\mathbb{G}$-adapted value processes S^i of names $i = 1, 2$, are given by

$$S_t^i = S_t^{i,0} 1_{t < \hat{\tau}_1} + S_t^{i,j}(\tau_j) 1_{\tau_j \leq t < \tau_i}, \quad t \geq 0, \ i, j = 1, 2, \ i \neq j,$$

where $S^0 = (S^{1,0}, S^{2,0})$ is the vector price process of the two names in absence of any default, governed by

$$dS_t^0 = \text{diag}(S_t^0)(b_t^0 dt + \sigma_t^0 dW_t),$$

$b^0 = (b^{1,0}, b^{2,0})$ is $\mathbb{F}$-adapted, σ^0 is the 2×2-diagonal $\mathbb{F}$-adapted matrix with diagonal diffusion coefficients $\sigma^{1,0} > 0$, $\sigma^{2,0} > 0$, and the indexed process $S^{i,j}(\theta_j)$, $\theta_j \in \mathbb{R}_+$, representing the price process of name i after the default of name j at time θ_j, is given by

$$\begin{aligned}
dS_t^{i,j}(\theta_j) &= dS_t^{i,j}(\theta_j)(b_t^{i,j}(\theta_j)dt + \sigma_t^{i,j}(\theta_j)dW_t^i), \quad t \geq \theta_j, \\
S_{\theta_j}^{i,j}(\theta_j) &= S_{\theta_j}^{i,0}.(1 + e^{i,j}),
\end{aligned}$$

where $e^{i,j}$ represents the proportional jump induced by the default of name j on name i, and assumed constant for simplicity and valued in $(-1, \infty)$. The coefficients $b^{i,0}$, $\sigma^{i,0} > 0$ are $\mathbb{F}$-adapted processes, and $b^{i,j}$, $\sigma^{i,j} > 0$ are in $\mathcal{O}_{\mathbb{F}}^1(\mathbb{R}_+)$.

The trading strategy of the investor is a $\mathbb{G}$-predictable measurable process α representing the fraction of wealth invested in the two names. It is then decomposed in four components: the first component α^0 is a pair of $\mathbb{F}$-predictable processes representing the fraction invested in the two names before any default, the second component $\alpha^{1,1}$ is an indexed $\mathbb{F}$-predictable process representing the fraction invested in the name 2 when the name 1 defaults, the third

component $\alpha^{1,2}$ is an indexed $\mathbb{F}$-predictable process representing the fraction invested in the name 1 when the name 2 defaults, and the fourth component is zero when both names default. The wealth process of the investor is then given by

$$X_t = X_t^0 1_{t<\hat{\tau}_1} + X_t^{1,1}(\tau_1)1_{\tau_1 \leq t < \tau_2} + X_t^{1,2}(\tau_2)1_{\tau_2 \leq t < \tau_1} + X_t^2(\tau_1, \tau_2)1_{t \geq \hat{\tau}_2},$$

where X^0 is the wealth process before any default, governed by

$$\begin{aligned} dX_t^0 &= X_t^0 (\alpha_t^0)' \mathrm{diag}(S_t^0)^{-1} dS_t^0 \\ &= X_t^0 (\alpha_t^0 . b_t^0 dt + (\alpha_t^0)' \sigma_t^0 dW_t), \end{aligned}$$

$X^{1,1}(\theta_1)$ is the wealth indexed process after default of name 1, governed by

$$\begin{aligned} dX_t^{1,1}(\theta_1) &= X_t^{1,1}(\theta_1)\alpha_t^{1,1}(\theta_1)\frac{dS_t^{2,1}(\theta_1)}{S_t^{2,1}(\theta_1)}, \quad t \geq \theta_1 \\ X_{\theta_1}^{1,1}(\theta_1) &= X_{\theta_1}^0 . (1 + \alpha_{\theta_1}^0 . (-1, e^{2,1})), \end{aligned}$$

$X^{1,2}(\theta_2)$ is the wealth indexed process after default of name 2, governed by

$$\begin{aligned} dX_t^{1,2}(\theta_2) &= X_t^{1,2}(\theta_2)\alpha_t^{1,2}(\theta_2)\frac{dS_t^{1,2}(\theta_2)}{S_t^{1,2}(\theta_2)}, \quad t \geq \theta_2 \\ X_{\theta_2}^{1,2}(\theta_2) &= X_{\theta_2}^0 . \big(1 + \alpha_{\theta_2}^0 . (e^{1,2}, -1)\big), \end{aligned}$$

and $X^2(\theta_1, \theta_2)$ is the wealth indexed process after both defaults, hence constant after $\theta_1 \vee \theta_2$, and then given by

$$X_t^2(\theta_1, \theta_2) = \begin{cases} X_{\theta_2}^{1,1}(\theta_1) . \big(1 - \alpha_{\theta_2}^{1,1}(\theta_1)\big), & \theta_1 \leq \theta_2 \leq t \\ X_{\theta_1}^{1,2}(\theta_2) . \big(1 - \alpha_{\theta_1}^{1,2}(\theta_2)\big), & \theta_2 < \theta_1 \leq t \end{cases}$$

In order to ensure that the wealth process is strictly positive, we assume that α^0 is valued in a closed subset $A_0 \subset \{a \in \mathbb{R}^2 : 1 + a.(-1, e^{2,1}) > 0, \text{ and } 1 + a.(e^{1,2}, -1) > 0\}$, and $\alpha^{1,1}$, $\alpha^{1,2}$ are valued respectively in closed subsets $A_{1,1}, A_{1,2} \subset (-\infty, 1)$.

We are next given a utility function U on $\mathbb{R}_+$, over a finite horizon T, and we consider the optimal investment problem

$$V_0(x) = \sup_{\alpha \in \mathcal{A}_{\mathbb{G}}} \mathbb{E}\big[U(X_T^{x,\alpha})\big]. \tag{6.8}$$

We use the $\mathbb{F}$-decomposition method of Section 5 for the resolution of (6.8). From Theorem 5.1, the value function V_0 is obtained via the following backward induction formula:

$$V_2(x, \theta_1, \theta_2) = U(x)\mathbb{E}\big[\gamma_T(\theta_1, \theta_2)\big|\mathcal{F}_{\theta_1 \vee \theta_2}\big] := U(x)\bar{\gamma}(\theta_1, \theta_2)$$

$$V_{1,1}(x, \theta_1) = \operatorname*{ess\,sup}_{\alpha^{1,1} \in \mathcal{A}_{\mathbb{F}}^{1,1}} \mathbb{E}\Big[U(X_T^{1,1,x})\gamma_T^{1,1}(\theta_1) + \int_{\theta_1}^T U\big(X_{\theta_2}^{1,1} . (1 - \alpha_{\theta_2}^{1,1})\big)\bar{\gamma}(\theta_1, \theta_2)d\theta_2 \Big|\mathcal{F}_{\theta_1}\Big]$$

$$V_{1,2}(x, \theta_2) = \operatorname*{ess\,sup}_{\alpha^{1,2} \in \mathcal{A}_{\mathbb{F}}^{1,2}} \mathbb{E}\Big[U(X_T^{1,2,x})\gamma_T^{1,2}(\theta_2) + \int_{\theta_2}^T U\big(X_{\theta_1}^{1,2} . (1 - \alpha_{\theta_1}^{1,2})\big)\bar{\gamma}(\theta_1, \theta_2)d\theta_1 \Big|\mathcal{F}_{\theta_1}\Big]$$

$$\begin{aligned} V_0(x) = \sup_{\alpha^0 \in \mathcal{A}_{\mathbb{F}}^0} \mathbb{E}\Big[&U(X_T^{0,x})\gamma_T^0 \\ &+ \int_0^T V_{1,1}\big(X_\theta^0 . (1 + \alpha_\theta^0 . (-1, e^{2,1})), \theta\big) + V_{1,2}\big(X_\theta^0 . (1 + \alpha_\theta^0 . (e^{1,2}, -1)), \theta\big) \, d\theta\Big]. \end{aligned}$$

In the sequel, we consider power utility functions

$$U(x) = \frac{1}{p}x^p, \quad x \geq 0, \ p < 1, \ p \neq 0,$$

and we use dynamic programming and BSDE methods in the $\mathbb{F}$-filtration to solve the above stochastic control problems. The value functions $V_{1,1}$ and $V_{1,2}$ are then in the form

$$V_{1,1}(x,\theta_1) = U(x)Y_{\theta_1}^{1,1}(\theta_1), \qquad V_{1,2}(x,\theta_2) = U(x)Y_{\theta_2}^{1,2}(\theta_2),$$

where $Y^{1,1}(\theta_1)$ and $Y^{1,2}(\theta_2)$ are solutions to the BSDEs:

$$Y_t^{1,1}(\theta_1) = \gamma_T^{1,1}(\theta_1) + \int_t^T f_{1,1}(r, Y_r^{1,1}(\theta_1), Z_r^{1,1}(\theta_1), \theta_1)dr - \int_t^T Z_r^{1,1}(\theta_1)dW_r^2,$$

$$Y_t^{1,2}(\theta_2) = \gamma_T^{1,2}(\theta_2) + \int_t^T f_{1,2}(r, Y_r^{1,2}(\theta_2), Z_r^{1,2}(\theta_2), \theta_2)dr, - \int_t^T Z_r^{1,2}(\theta_2)dW_r^1$$

with generators

$$f_{1,1}(t,y,z,\theta_1) = p \sup_{a \in A_{1,1}} \left[(b_t^{2,1}(\theta_1)y + \sigma_t^{2,1}(\theta_1)z)a - \frac{1-p}{2}y|\sigma_t^{2,1}(\theta_1)a|^2 \right.$$
$$\left. + \bar{\gamma}(\theta_1,t)\frac{(1-a)^p}{p} \right]$$

$$f_{1,2}(t,y,z,\theta_2) = p \sup_{a \in A_{1,2}} \left[(b_t^{1,2}(\theta_2)y + \sigma_t^{1,2}(\theta_2)z)a - \frac{1-p}{2}y|\sigma_t^{1,2}(\theta_2)a|^2 \right.$$
$$\left. + \bar{\gamma}(t,\theta_2)\frac{(1-a)^p}{p} \right].$$

Finally, we have

$$V_0(x) = U(x)Y_0,$$

where Y^0 is the solution to the BSDE

$$Y_t^0 = \gamma_T^0 + \int_t^T f_0(r, Y_r^0, Z_r^0)dr - \int_t^T Z_r^0.dW_r,$$

with a generator

$$f_0(t,y,z) = p \sup_{a \in A_0} \left[(yb_t^0 + \sigma_t^0 z).a - \frac{1-p}{2}y|\sigma_t^0 a|^2 \right.$$
$$\left. + Y_t^{1,1}(t)\frac{(1+a.(-1,e^{2,1}))^p}{p} + Y_t^{1,2}(t)\frac{(1+a.(e^{1,2},-1))^p}{p} \right].$$

The details and rigorous mathematical treatment of the above derivation are studied in [13], where we prove the existence and uniqueness of the solutions to these BSDEs, and that they are indeed related to the original value functions of our optimal investment problem.

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